12 Complex Numbers and Functions Fall 2003

Complex numbers were invented in the context of solutions of algebraic equations. Some quadratic equations, in the form $ax^2 + bx + c = 0$, have two solutions that are ordinary (i.e., real) numbers, that is, two values of x that satisfy the equation. But some quadratic equations have *no* solutions in terms of ordinary numbers. Two simple examples are

$$x^2 + 1 = 0 \qquad \text{and} \qquad x^2 - 2x + 2 = 0.$$

In the first case, elementary algebra gives $x = \pm\sqrt{-1}$; in the second, the quadratic formula gives $x = 1 \pm \sqrt{-1}$. There is no real number whose square is -1. So we have two choices: Either the above equations have no solutions, or else we need to broaden our concept of *number*.

Complex Numbers

The second choice is the more useful one. We define a more general class of numbers, called *complex numbers*, and define algebraic operations and properties for them. First we define an imaginary unit, denoted by i, defined as the square root of -1: $i = \sqrt{-1}$ and $i^2 = -1$. (In Maple the imaginary unit is denoted by capital I; in a-c circuit analysis it is usually denoted by j.) The imaginary unit can be multiplied by a real number, such as a or b. By definition, this product obeys the commutative and distributive rules: for any real numbers a and b, $ia = ai$ and $(a + b)i = ai + bi$. Any real number multiplied by i, such as ai, is called an *imaginary number*.

If $z = a + ib$, where a and b are real numbers (i.e., not imaginary), then z is called a *complex number*. The *real* part of z is a, and the *imaginary* part of z is b (not ib).

Next we *define* addition and multiplication of two complex numbers z_1 and z_2, following these general requirements:

(1) The definitions must be given in terms of the definitions of corresponding operations for *real* numbers;

(2) When both of the complex numbers happen to be real (i.e., have zero imaginary part), the definitions must reduce to the definitions for real numbers;

(3) The operations must obey the same rules (associative, commutative, and distributive rules) as the corresponding operations with real numbers.

Following these principles, we *define* addition and multiplication of two complex numbers z_1 and z_2 as follows: If a_1, a_2, b_1, and b_2 are real, and

$$z_1 = a_1 + ib_1 \qquad \text{and} \qquad z_2 = a_2 + ib_2, \qquad \text{then}$$

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2), \tag{1}$$

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1). \tag{2}$$

With these definitions, it is easy to prove that the sum obeys the associative and commutative laws:

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \qquad \text{and} \qquad z_1 + z_2 = z_2 + z_1, \tag{3}$$

and that the product obeys the associative, commutative, and distributive laws:

$$z_1(z_2 z_3) = (z_1 z_2)z_3, \qquad z_1 z_2 = z_2 z_1, \qquad \text{and} \qquad z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3. \tag{4}$$

We leave the proofs of these statements as exercises.

The *absolute value* of z, also called the *magnitude* of z, denoted by $|z|$, is defined as

$$|z| = \sqrt{a^2 + b^2}. \tag{5}$$

The *complex conjugate* of z, denoted by z^*, is defined as

$$z^* = a - i\,b. \tag{6}$$

That is, the complex conjugate of z has the same real part as z, but its imaginary part has the opposite sign.

From these definitions, $zz^* = |z|^2$. This quantity is always real and non-negative.

For any two complex numbers z_1 and z_2, $|z_1 z_2| = |z_1| \cdot |z_2|$ and $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$.

To find the real and imaginary parts of the complex number $\dfrac{1}{a+ib}$, where a and b are real, multiply numerator and denominator by the complex conjugate of the denominator $(a - ib)$.

$$\frac{1}{a+ib} = \frac{1}{a+ib}\,\frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} + i\,\frac{-b}{a^2+b^2}. \tag{7}$$

Thus the real part of $\dfrac{1}{a+ib}$ is $\dfrac{a}{a^2+b^2}$, and its imaginary part is $\dfrac{-b}{a^2+b^2}$. This process is called *rationalizing*. Also note that $1/i = -i$.

For any complex quantity C, the notations $\mathrm{Re}(C)$ and $\mathrm{Im}(C)$ are often (including in Maple) used to denote the real and imaginary parts, of C. In the above example,

$$\mathrm{Re}\left(\frac{1}{a+ib}\right) = \frac{a}{a^2+b^2} \qquad \text{and} \qquad \mathrm{Im}\left(\frac{1}{a+ib}\right) = \frac{-b}{a^2+b^2}. \tag{8}$$

If a and b are real, the complex conjugate of $\dfrac{1}{a+ib}$ is $\dfrac{1}{a-ib}$. More generally, in any complex algebraic expression where all the symbols represent real quantities, the complex conjugate can be obtained by replacing i everywhere by $(-i)$. We invite you to prove these statements.

Complex Functions

When the real and imaginary parts of a complex quantity are variables, we call the quantity a *complex variable*. We'll use the notation $z = x + iy$ for a complex variable z with real part x and imaginary part y, where x and y are both *real*.

We'll define the sine, cosine, and exponential function of a complex variable, following the same requirements as for the definitions of complex numbers:

(1) The definitions must be given in terms of real functions of real variables.

(2) When the complex variable z happens to be real (i.e., $y = 0$), the definition must reduce to the definition of the corresponding real function of a real variable.

We begin with the exponential function of a complex variable $z = x + iy$. We *define* e^z:

$$e^z = \exp(z) = e^x(\cos y + i\sin y). \tag{9}$$

We note that this definition satisfies the above requirements; e^x, $\sin x$, and $\cos x$ are all real functions of a real variable, and when $y = 0$, $e^z = e^x$. It is also easy to show that the exponential function defined in this way satisfies the law of exponents:

$$e^{(z_1+z_2)} = e^{z_1}e^{z_2}. \tag{10}$$

We leave the proof of this statement as an exercise.

Note that when z is purely imaginary (i.e., when $x = 0$), the above definition gives

$$e^{iy} = \cos y + i\sin y. \tag{11}$$

This equation is called *Euler's formula*, and it follows very simply from our general definition of the exponential function of z. Euler's formula is often "derived" by combining the Taylor series expansions for $\cos y$ and $i\sin y$. This derivation is of doubtful validity unless it is preceded by some discussion of convergence of series of complex quantities. So we prefer to avoid this approach.

Next we consider the sine and cosine of a complex variable. The following definitions may look a little strange, but we'll show that they are consistent with the above requirements. We *define* $\sin z$ and $\cos z$ as follows:

$$\sin z = \frac{\left(e^{iz} - e^{-iz}\right)}{2i}, \qquad \cos z = \frac{\left(e^{iz} + e^{-iz}\right)}{2}. \tag{12}$$

We have to show that these definitions reduce to the familiar ones when z is real, that is, when $y = 0$. To do this, we first use Eq. (9) to express e^{iz} and e^{-iz} in terms of real functions, also using the fact that $\sin(-x) = -\sin(x)$. (Note that the real part of $iz = i(x + iy)$ is $(-y)$ and the imaginary part is x.)

$$e^{iz} = e^{(-y+ix)} = e^{-y}(\cos x + i\sin x) \quad \text{and} \quad e^{-iz} = e^{(y-ix)} = e^{y}(\cos x - i\sin x) \tag{13}$$

When these expressions are substituted into the first of Eqs. (12), we get

$$\sin z = \frac{1}{2i}\left[\left(e^{-y} - e^{y}\right)\cos x + i\left(e^{-y} + e^{y}\right)\sin x\right] \tag{14}$$

When z is real, $y = 0$ and the right side of Eq. (14) reduces to $\sin x$, and Requirement (2) (page 12-3) is satisfied. A similar calculation can be made for $\cos z$ to show that it reduces to $\cos x$ when z is real.

As a dividend, we note that when z is pure imaginary, $x = 0$. In that case, Eq. (13) and the corresponding equation for $\cos z$ give

$$\sin i y = \frac{e^{-y} - e^{y}}{2i} = i \sinh y, \qquad \cos i y = \frac{e^{y} + e^{-y}}{2} = \cosh y. \tag{15}$$

There are many other remarkable relationships among the trigonometric and hyperbolic functions.

The Complex Plane

A complex number can be represented as a point in the x-y plane, with the real part plotted along the x axis and the imaginary part along the y axis. This point can also be represented in terms of its polar coordinates (r, θ), where

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = x + iy = r(\cos\theta + i\sin\theta), \tag{16}$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\frac{y}{x}, \tag{17}$$

The angle θ is called the *argument* or *phase* of z, and r is its *absolute value*. By convention, r is always the positive root, and θ is taken either in the range from $-\pi$ to π or from 0 to 2π. Note that there is an ambiguity in the definition of θ in Eq. (17), since there are two possible angles for any given values of x and y. For example, if $x = 1$ and $y = -1$, Eq. (17) gives the two values $\theta = 3\pi/4$ and $\theta = -\pi/4$. Comparison with Eq. (16) and the values of x and y shows that in this case the second value is the correct one.

With Euler's formula, Eq. (16) can also be written as

$$z = re^{i\theta} \tag{18}$$

Any complex number z can be written in this form, called the *polar form* of z.

Also, for any complex number z, $\dfrac{1}{z} = \dfrac{1}{r}e^{-i\theta}$. This shows that the argument of $1/z$ is the negative of the argument of z.

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$. In a-c circuit analysis, θ plays the role of a phase angle; this important result shows that when two complex quantities are multiplied (such as a current I and an impedance Z), the phases *add*.

Maple Commands

Here are a few useful Maple commands for complex numbers:

abs(z); absolute value of z

argument(z); polar angle of z

Re(z); real part of z

Im(z); imaginary part of z

conjugate(z); complex conjugate of z

convert(z, polar); converts z to polar (r, θ) representation

evalc(...); is used in conjunction with the above commands, to separate the real and imaginary parts of a complex expression. When evalc is used, Maple assumes that all symbolic quantities (such as a and b in the examples below) are real quantities, unless they are explicitly defined otherwise (such as z in the examples).

Examples:

```
z := a + I*b;
```

$z := a + ib;$

```
evalc(Re(z));
```

a

```
evalc(abs(z));
```

$\sqrt{a^2 + b^2}$

```
evalc(conjugate(z));
```

$a - ib$

```
evalc(1/z);
```

$$\frac{a}{a^2 + b^2} + \frac{-Ib}{a^2 + b^2}$$

```
evalc(Im(1/z));
```

$$\frac{-b}{a^2 + b^2}$$