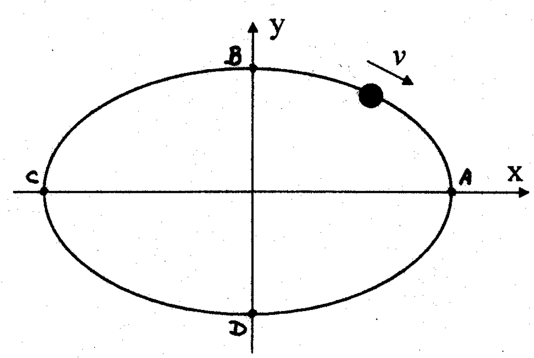
Quiz #1 Solutions

Problem: Consider a particle that moves along an ellipse, $x^2/a^2+y^2/b^2=1$ ($a>b$). Let the speed of the particle be a constant v.

1. (10 points) State and qualitatively justify where on the elliptical path the magnitude of the particle's acceleration achieves its maximum and minimum.
2. (30 points) Compute the maximum and minimum of the magnitude of the particle's acceleration. To do so, you will need the radius of curvature of the elliptical path. Express the ellipse in terms of two parametric equations: $x=x(t)$, $y=y(t)$, and then compute the radius of curvature of using the formula: $\rho=\{[x'(t)]^2+[y'(t)]^2\}^{3/2}/[x'(t)y''(t)-x''(t)y'(t)]$. Note that the parameter t here is purely geometric in nature, which may or may not be the time variable.



Solution:

1. Since $v=\text{const}$, the tangential acceleration is zero everywhere on the path. Thus, $a=a_n$ where $a_n=\frac{v^2}{\rho}$ is the normal acceleration, and ρ is the radius of curvature.

   Since ρ achieves its minimum at points A & C, a & a_n achieve their maximum at A & C. On the other hand, ρ is maximal at B & D, and hence a is minimal at these points.

2. Parametric equations: $x=a\cos t$, $y=b\sin t$.

   $\dot{x}=-a\sin t$, $\dot{y}=b\cos t$, $\ddot{x}=-a\cos t$, $\ddot{y}=-b\sin t$.

   $$\rho=\frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}}=\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab\sin^2 t+ab\cos^2 t}=\frac{1}{ab}(a^2\sin^2 t+b^2\cos^2 t)^{3/2}$$

At point A, $t = 0$ (hence $x = a$, $y = 0$) :

$$\rho_{min} = \frac{(b^2)^{3/2}}{ab} = \frac{b^2}{a} \Rightarrow a_{max} = \frac{a v^2}{b^2}$$

At point B, $t = \pi/2$ (i.e. $x = 0$, $y = b$)

$$\rho_{max} = \frac{(a^2)^{3/2}}{ab} = \frac{a^2}{b} \Rightarrow a_{min} = \frac{b v^2}{a^2}$$