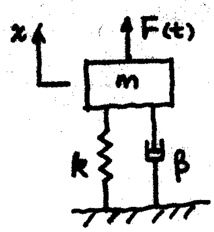
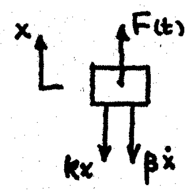
Equation of Motion





$$m\ddot{x} = -kx - \beta\dot{x} + F(t)$$

i.e. $m\ddot{x} + \beta\dot{x} + kx = F(t)$

Introduce two parameters:

$\omega_n = \sqrt{\frac{k}{m}}$ (natural frequency)

$\zeta = \frac{\beta}{2\sqrt{mk}}$ (damping ratio)

Equation of motion becomes

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F}{m}} \qquad (1)$$

This equation can be solved with initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = v_0 .$$

Free Oscillations : F=0

$$\begin{cases} \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \\ x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases} \qquad (2)$$

Assume the solution to be of the form $x = Ce^{\lambda t}$. Then

$$(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2)Ce^{\lambda t} = 0$$

For a nontrivial solution $C \neq 0$. Hence,

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

This is the characteristic equation for the ODE, and can be solved to give

$$\lambda = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n . \qquad (3)$$

<u>Case 1:</u> $\zeta > 1$. Eq. (3) gives two distinct real roots.

$$x = Ce^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t} + De^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t}$$

To determine C & D, apply the I.C.'s:

$$x(0) = C + D = x_0$$

$$\dot{x}(0) = -(\zeta - \sqrt{\zeta^2 - 1})\omega_n C - (\zeta + \sqrt{\zeta^2 - 1})\omega_n D = v_0$$

$$\Rightarrow \qquad C = \frac{x_0(\zeta + \sqrt{\zeta^2 - 1})\omega_n + v_0}{2\sqrt{\zeta^2 - 1}\,\omega_n}$$

$$D = -\frac{x_0(\zeta - \sqrt{\zeta^2 - 1})\omega_n + v_0}{2\sqrt{\zeta^2 - 1}\,\omega_n}$$

The solution therefore is:

$$x(t) = \frac{1}{2\sqrt{\zeta^2-1}\,\omega_n}\Big\{\left[x_0(\zeta+\sqrt{\zeta^2-1})\omega_n + v_0\right]e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t}$$
$$-\left[x_0(\zeta-\sqrt{\zeta^2-1})\omega_n + v_0\right]e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t}\Big\}$$

The solution is plotted in the figure below (for $v_0 = 0$).

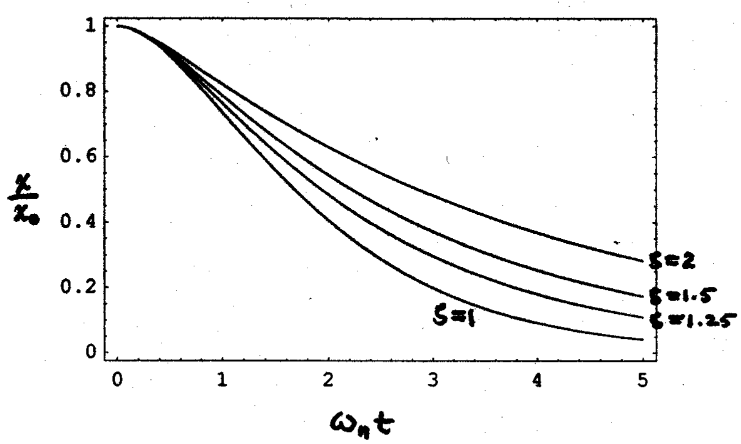


Fig. 1: Free oscillation for $\zeta \geq 1$.

A system with $\zeta > 1$ is said to be overdamped.

Case 2: $\zeta = 1$. Eq. (3) gives two identical roots $\lambda = -\omega_n$.

The solution to Eq. (2) now takes the form

$$x = Ce^{-\omega_n t} + Dte^{-\omega_n t}.$$

Applying the I.C.'s to find C and D, we obtain

$$x = [x_0 + (v_0 + x_0\omega_n)t]e^{-\omega_n t},$$

which is also plotted in Fig. 1. In this case the system is critically damped.

Case 3: $\zeta < 1$ (the system is underdamped). Eq. (3) gives

$$\lambda = -\zeta\omega_n \pm \sqrt{1-\zeta^2}\,\omega_n i$$

The solution to Eq. (2) is then of the form

$$x = e^{-\zeta\omega_n t}\left(C\cos(\sqrt{1-\zeta^2}\,\omega_n t) + D\sin(\sqrt{1-\zeta^2}\,\omega_n t)\right).$$

We can again determine C and D from the I.C.'s. Then

$$x = e^{-\zeta\omega_n t}\left\{x_0\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\zeta\omega_n x_0 + v_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

The solution is oscillatory (figure below for $v_0 = 0$).

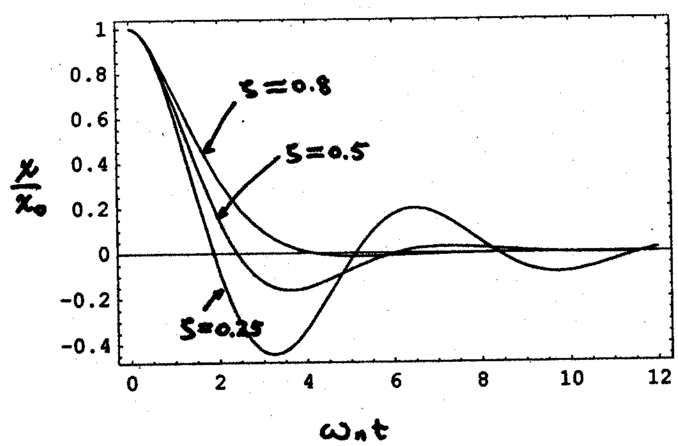


Fig. 2. Free oscillation of an underdamped system

From Figs. 1 and 2, we observe oscillations as t becomes sufficiently large, x decays to zero.

Forced Oscillations : F = const.

Now, we need to solve the nonhomogeneous ODE (1). The strategy is to combine a particular solution to (1), which does not necessarily satisfy the I.C.'s, with the general solution to the homogeneous ODE obtained by setting F to zero. Here we focus on the underdamped case, which is the most common in practice.

The general solution to (1) takes the form

$$x = e^{-\zeta\omega_n t}\left(C\cos\sqrt{1-\zeta^2}\,\omega_n t + D\sin\sqrt{1-\zeta^2}\,\omega_n t\right) + \frac{F}{m\omega_n^2}. \quad (4)$$

Note that $\frac{F}{m\omega_n^2} = \frac{F}{k}$ is a particular solution to (1).

Substituting (4) into the I.C.'s, we can determine C and D, and

$$x = \underbrace{e^{-\zeta\omega_n t}\left\{\left(x_0-\frac{F}{k}\right)\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\zeta\omega_n\left(x_0-\frac{F}{k}\right)+v_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}}_{\text{Transient response}} + \underbrace{\frac{F}{k}}_{\text{steady-state response}}$$

As shown in Fig. 3, the transient responce, which depends on the I.C.'s, dies out as $t\to\infty$. The only remaining part of the solution is the steady-state response $\frac{F}{k}$, which is I.C.-independent. Also note that F/k is just the static deflection of the spring since F = const.

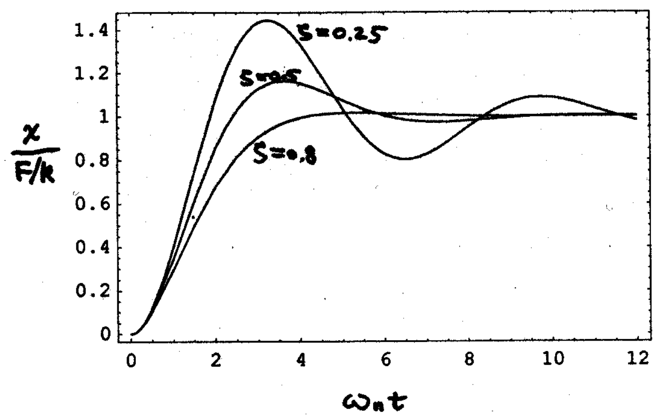


Fig. 3. Forced oscillation of an underdamped system

($F = \text{const}$, $x_0 = 0$, $v_0 = 0$)

Forced Oscillations : $F = F_{max} \cos\omega t$

The nonhomogeneous ODE can be solved in a way very similar to the case $F = \text{const}$. For example, if the system is underdamped, the general solution to this periodic forcing problem is

$$x = \underbrace{e^{-\zeta\omega_n t}\left\{C\cos\sqrt{1-\zeta^2}\,\omega_n t + D\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}}_{\text{Transient response}} + \underbrace{A\cos(\omega t + \phi)}_{\text{periodic response}} \quad (5)$$

where C, D, A and ϕ are to be determined. Since the transient response dies out eventually, we are more interested in the steady-state response, which mathematically is a particular solution to ODE (1).

Let us determine A & ϕ. Substituting $x = A\cos(\omega t + \phi)$ into (1) gives

$$(\omega_n^2 - \omega^2) A \cos(\omega t + \phi) - 2\zeta \omega_n \omega A \sin(\omega t + \phi)$$

$$= \frac{F_{max}}{m} \cos \omega t$$

$$= \frac{F_{max}}{m} \cos\phi \cos(\omega t + \phi) + \frac{F_{max}}{m} \sin\phi \sin(\omega t + \phi) .$$

Equating coefficients of $\cos(\omega t + \phi)$ and $\sin(\omega t + \phi)$ gives

$$\begin{cases} (\omega_n^2 - \omega^2) A = \frac{F_{max}}{m} \cos\phi \\ -2\zeta \omega_n \omega A = \frac{F_{max}}{m} \sin\phi \end{cases} \qquad (6)$$

Thus,

$$\tan\phi = \frac{2\zeta}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} , \qquad (7a)$$

and

$$A = \frac{F_{max}}{m\omega_n^2} \frac{1}{\sqrt{\left[\left(\frac{\omega}{\omega_n}\right)^2 - 1\right]^2 + 4\zeta^2 \left(\frac{\omega}{\omega_n}\right)^2}} . \qquad (7b)$$

Note that $\frac{F_{max}}{m\omega_n^2} = \frac{F_{max}}{k}$, and the expressions given by (7a) and (7b) are valid for overdamped, critically damped and underdamped systems. The magnitude A is plotted in Fig. 4 as a function of the frequency ratio $\frac{\omega}{\omega_n}$.

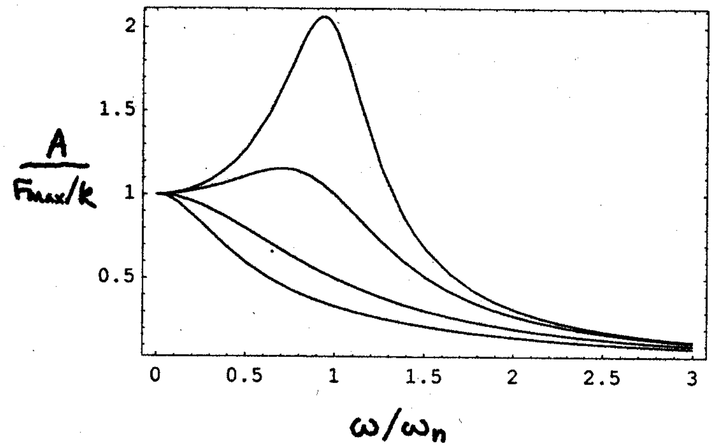


Fig. 4. Magnitude of steady-state response to periodic forcing.

From Eq (7b) and Fig. 4, we can make the following observations.

① For an underdamped system ($\zeta<1$), A achieves a maximum at $\omega_r = \sqrt{1-2\zeta^2}\,\omega_n$ if $\zeta < \frac{1}{\sqrt{2}}$ (i.e., when the system is suffiently underdamped.). This phenomenon is called <u>resonance</u>, and at the resonace frequency ω_r, $A(\omega_r)/A(0) = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$.

② For all values of ζ, $A \to 0$ as $\omega \to \infty$. That is, the system ceases to respond to the forcing at sufficiently high frequencies. We define a <u>cutoff frequency</u> by $A(\omega_c)/A(0) = \frac{1}{\sqrt{2}}$. From (7b), $\omega_c = \omega_n\sqrt{1-2\zeta^2+\sqrt{(1-2\zeta^2)^2+1}}$.