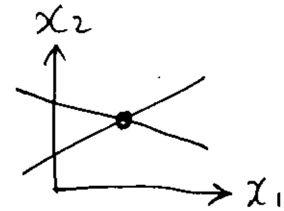
2.7 Systems of Nonlinear Equations

The most general equation form of root finding

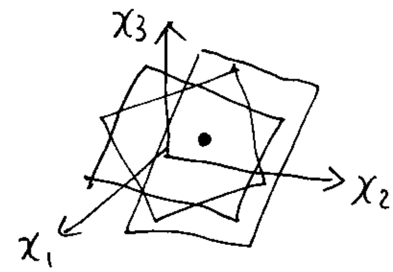
$$\begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0 \\ f_2(x_1, x_2, \cdots, x_n) = 0 \\ \quad \vdots \\ f_n(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

Linear functions

e.g., $\begin{cases} ax_1 + bx_2 + c = 0 \\ dx_1 + ex_2 + f = 0 \end{cases}$



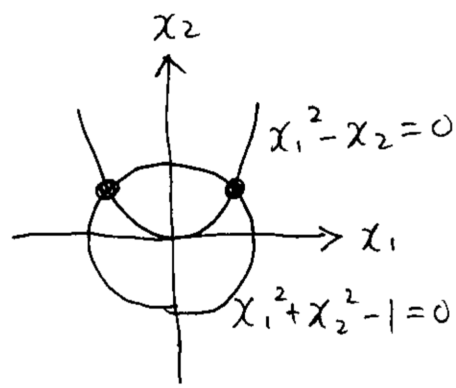
$$\begin{cases} ax_1 + bx_2 + cx_3 + d = 0 \\ ex_1 + fx_2 + gx_3 + h = 0 \\ ix_1 + jx_2 + kx_3 + l = 0 \end{cases}$$



$\Rightarrow \begin{bmatrix} & A & \end{bmatrix} \begin{Bmatrix} x \end{Bmatrix} = \begin{Bmatrix} b \end{Bmatrix}$: linear algebraic eq.

Nonlinear functions

e.g., $\begin{cases} x_1^2 - x_2 = 0 \\ x_1^2 + x_2^2 - 1 = 0 \end{cases}$



Newton-Raphson for Systems of Non-Linear Eq.

Apply Two variable Taylor Series expansion.
(multi)

$f(x_{i+1}, y_{i+1})$

$$= \sum_{k=0}^{n} \frac{1}{k!} \left((x_{i+1} - x_i)\frac{\partial}{\partial x} + (y_{i+1} - y_i)\frac{\partial}{\partial y} \right)^k f(x_i, y_i) + \underbrace{R_n}_{\text{truncation error}}$$

$$R_n = \frac{1}{(n+1)!} \left((x_{i+1} - x_i)\frac{\partial}{\partial x} + (y_{i+1} - y_i)\frac{\partial}{\partial y} \right)^{n+1} f(\xi_x, \xi_y)$$

where

$$x_i < x_\xi < x_{i+1}, \quad y_i < y_\xi < y_{\xi+1}$$

Sample problem

$$\begin{cases} u(x,y) = x^2 - y = 0 \\ v(x,y) = x^2 + y^2 - 1 = 0 \end{cases}$$

1st order Taylor series. approximation

$$\begin{cases} u_{i+1} = u_i + (x_{i+1} - x_i)\dfrac{\partial u_i}{\partial x} + (y_{i+1} - y_i)\dfrac{\partial u_i}{\partial y} = 0 \\[2ex] v_{i+1} = v_i + (x_{i+1} - x_i)\dfrac{\partial v_i}{\partial x} + (y_{i+1} - y_i)\dfrac{\partial v_i}{\partial y} = 0 \end{cases}$$

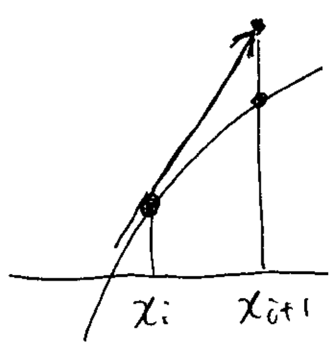
Solve the simultaneous linear eq's for x_{i+1} and y_{i+1}

$$x_{i+1} = x_i - \frac{u_i \dfrac{\partial v_i}{\partial y} - v_i \dfrac{\partial u_i}{\partial y}}{\dfrac{\partial u_i}{\partial x}\dfrac{\partial v_i}{\partial y} - \dfrac{\partial u_i}{\partial y}\dfrac{\partial v_i}{\partial x}}$$

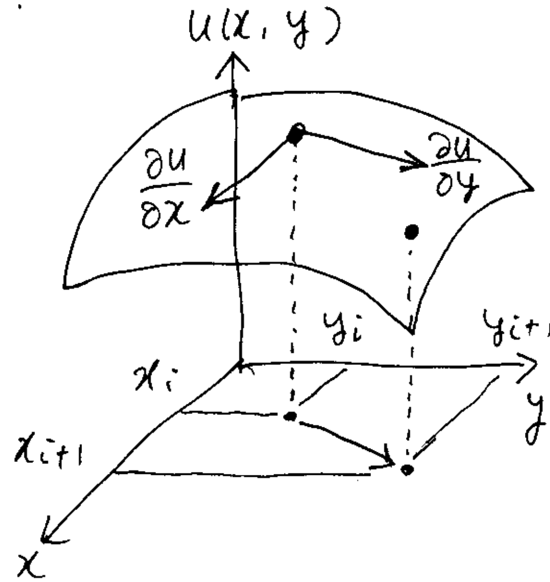
$$y_{i+1} = y_i - \frac{v_i \dfrac{\partial u_i}{\partial x} - u_i \dfrac{\partial v_i}{\partial x}}{\dfrac{\partial u_i}{\partial x}\dfrac{\partial v_i}{\partial y} - \dfrac{\partial u_i}{\partial y}\dfrac{\partial v_i}{\partial x}}$$

The determinant of the Jacobian of the system.

Geometric interpretation.



$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$



$$u = x^2 - y \qquad v = x^2 + y^2 - 1$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -1$$

$$\frac{\partial v}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2y.$$

$$\begin{cases} x_{i+1} = x_i - \dfrac{(x_i^2 - y_i)\cdot 2y_i + (x_i^2 + y_i^2 - 1)}{4x_i y_i + 2x_i} \\[2ex] y_{i+1} = y_i - \dfrac{(x_i^2 + y_i^2 - 1)\cdot 2x_i - (x_i^2 - y_i)\cdot 2x_i}{4x_i y_i + 2x_i} \end{cases}$$

$\Uparrow$

Iterate this until it converges.

Note: This has the quadratic convergence rate.

Note: Use the modified Newton-Raphson for multiple roots.

e.g.)