Department of Mathematical Sciences
Carnegie Mellon University

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21-256 Mulivariate Analysis and Approximation

# Optimization with Constraints: outline of solution method

Notation:

$$x = (x_1, x_2, \ldots, x_n).$$

Statement of problem (one constraint):

Minimize (or maximize): $f(x)$

Subject to: $g(x) = c$

**Step 1:** Find all critical points for $g$ such that $g(x) = c$. In other words, find all solutions to

$$\nabla g(x) = 0;$$
$$g(x) = c.$$

**Step 2:** Find all critical points for the Lagrangian

$$L(x; \lambda) = f(x) + \lambda(c - g(x)).$$

In other words, find all solutions to

$$\nabla f(x) = \lambda \nabla g(x);$$
$$g(x) = c.$$

**Step 3:**
- If only one point is found from steps 1 and 2, then assume this is the solution.
- If more then one point is found from steps 1 and 2, then plug these points into $f$ and compare the resulting values:
  * If all points yield the same value for $f$, then assume the points found are all solutions.
  * Otherwise, the largest value obtained is the maximum for $f$ and the smallest value obtained for $f$ is the minimum.

Notation:

$$x = (x_1, x_2, \ldots, x_n);$$
$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Statement of problem (multiple constraints):
Minimize (or maximize): $f(x)$
Subject to: $g^1(x) = c_1,\ g^2(x) = c_2,\ \ldots,\ g^m(x) = c_m$

**Step 1:** Find all points such that $g^i(x) = c_i$ for each $i = 1, 2, \ldots, m$ and

$$\{\nabla g^1(x), \nabla g^2(x), \ldots, \nabla g^m(x)\}$$

is a linearly dependent set of vectors. In other words, find all point $x$ such that $g^i(x) = c_i$ for each $i = 1, 2, \ldots, m$ and there is a nontrivial solution to

$$a_1 \nabla g^1(x) + a_2 \nabla g^2(x) + \cdots + a_m \nabla g^m(x) = 0.$$

(By nontrivial, we mean that at least one of the $a$'s is non-zero.)

**Step 2:** Find all critical points for the Lagrangian

$$L(x; \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i (c_i - g^i(x)).$$

In other words, find all solutions to

$$\begin{aligned} \nabla f(x) &= \sum_{i=1}^{m} \lambda_i \nabla g^i(x); \\ g^1(x) &= c_1; \\ g^2(x) &= c_2; \\ &\vdots \\ g^m(x) &= c_m. \end{aligned}$$

(Note that the above is a system of $n + m$ equations.)

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  - If all points yield the same value for $f$, then assume the points found are all solutions.
  - Otherwise, the largest value obtained is the maximum for $f$ and the smallest value obtained for $f$ is the minimum.