21-228 Week 8 notes

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1 Law of Total Probability

Theorem 1.1. *Let $B_1, B_2, \ldots, B_n$ be pairwise disjoint events which partition Ω, for any other event A,*

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

Proof. Exercise. $\square$

Example:

Suppose we have two crooked dice, so that if the outcome of the first is X then the outcome of the second Y satisfies Y is equally likely to be $X-1, X, X+1$ (If $X = 1$ or $X = 6$, then instead we have two equallylikely values)

What is the probability that $X = Y$, if $P(X = i) = \frac{1}{6}$ for each i?

2 Random Variables

Now a new topic. Consider a probability space Ω. A *random variable* takes each outcome and assignes to it a real number. Thus, we have:

Definition 2.1. *A <u>random variable</u> is a function $X : \omega \to \mathbb{R}$.*

Definition 2.2. *Let $A \subseteq R$, X a random variable. Then* $\Pr(X \in A) = \Pr\{\omega : X(\omega) \in A\}$.

2.1 Examples

2.1.1 Two Dice

Consider rolling two fair dice. Then Ω is simply $[6]^2$. Let $X(\omega) = x_1 + x_2$ – that is, the sum of the numbers on the dice. Then what is $\Pr(X = k)$ for various k?

2.1.2 Tossing Coins

Suppose we toss n (distinguishable) loaded coins, where each coin has a probability p of coming up heads. Then $\Omega = \{H, T\}^n$, and $P(\omega) = p^k(1-p)^{n-k}$ if there are k heads in ω. Let $X(\omega)$ be the number of heads in ω.

There are $\binom{n}{k}$ ways of getting exactly k heads, each of which has probability $p^k(1-p)^{n-k}$ chance of happening. Therefore, $\Pr(X = k)$ is $\binom{n}{k}p^k(1-p)^{n-k}$.

This is a very common random variable. Notice that there are two parameters in the problem – the number of coins, n, and the probability of heads, p. Therefore, we call X the *binary randim variable* $B_{n,p}$.

2.1.3 Poisson Variable

Let $\Omega = \mathbb{N}$, and let X simply be the identity function. Let λ be a real number.

Suppose that $P(\omega = k) = \frac{\lambda^k e^{-\lambda}}{k!}$.

The resulting random variable is then called the *Poisson Random Variable*, and is denoted $Po(\lambda)$ (λ being the parameter – we will see later that λ has an important property).

Poisson variables and Binomial variables have an interesting relationship. The idea is that we want to model a situaton that, no matter how many trails we make, we "expect" (will formally define expectation shortly) to get the same number of successes. Thus, if we make very many trials, the probability of success is low, but we have a small number of trials, the probability of success is high.

Suppose that k, λ are fixed, and consider the binomial random variable $B(n, \lambda/n)$. Recall that $f(n) \approx g(n)$ iff $\lim_{n\to\infty} f(n)/g(n) = 1$.

Theorem 2.3.

$$\lim_{n\to\infty} P(B_{n,\lambda/n} = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Proof. First, we show that $\binom{n}{k} \approx \frac{n^k}{k!}$, for a fixed k.

$$
\begin{aligned}
\frac{n^k}{k!} &\geq \binom{n}{k} \\
&= \frac{n^k}{k!}(\frac{n-1}{n})(\frac{n-2}{n})\dots(\frac{n-k+1}{n}) \\
&= \frac{n^k}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\dots(1-\frac{k-1}{n} \\
&\geq \frac{n^k}{k!}(1-\frac{k(k-1)}{2n})
\end{aligned}
$$

The last line can be proven by inducton on k. Since we will let $n \to \infty$, the right factor in the last line above will tend to 1.

Now that we know $\binom{n}{k} \approx \frac{n^k}{k!}$, we have:

$$
\begin{aligned}
\lim_{n\to\infty} P(B_{n,\lambda/'n} = k) &= \lim_{n\to\infty} \binom{n}{k}(\frac{\lambda}{n})^k(1-\frac{\lambda}{n})^{n-k} \\
&= \frac{\lambda^k e^{-\lambda}}{k!}
\end{aligned}
$$

The last line follows from the approximation we made for $\binom{n}{k}$. □

2.2 Expectation of a random variable

Definition 2.4 (Expected Value). *If X is a random variable, then the expected value for X is given by*

$$E(X) = \sum_{k\in\mathbb{R}} kP(Z = k)$$

Of course, we only need to consider those k that Z takes with nonzero probability.

2.2.1 Two Dice

Suppose we toss two fair dice, and let Z be the sum of the two values. What is $E(Z)$?

2.2.2 Trials until success

Suppose we flip a coin with probability of heads p repeatedly until we get heads. Let X be the number of flips required. What is $E(Z)$?

2.2.3 Binomial Distribution

$$\begin{aligned}
E(B_{n,p}) &= \sum_{k=0}^{n} k \binom{n}{K} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} n \binom{n-1}{k-1} p^k (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \\
&= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} \\
&= np(p + (1-p))^{n-1} \\
&= np
\end{aligned}$$

The second-to-last line follows from the bimomial theorem – that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

2.2.4 Additivity Rule

Theorem 2.5 (Additivity Rule). *If X, Y are random variables on a probability space Ω, then $E(X+Y) = E(X) + E(Y)$*

Proof.

$$\begin{aligned}
E(X) + E(Y) &= \sum_{k \in \mathbb{R}} k \Pr(X = k) + \sum_{k \in \mathbb{R}} k \Pr(Y = k) \\
&= \sum_{k \in \mathbb{R}} k \sum_{\omega \in \Omega : X(\omega) = k} \Pr(\omega) + k \sum_{\omega \in \Omega : Y(\omega) = k} \Pr(\omega) \\
&= \sum_{\omega \in \Omega} X(\omega) \Pr(\omega) + \sum_{\omega \in \Omega} Y(\omega) \Pr(\omega) \\
&= \sum_{\omega \in \Omega} (X(\omega) + Y(\omega)) \Pr(\omega) \\
&= \sum_{\omega \in \Omega} (X+Y)(\omega) \Pr(\omega) \\
&= \sum_{k \in \mathbb{R}} k \sum_{\omega \in \Omega : (X+Y)(\omega) = k} \Pr(\omega) \\
&= \sum_{k \in \mathbb{R}} k \Pr(X + Y = k) \\
&= E(X+Y)
\end{aligned}$$

$\square$

This can be used to find a much simpler proof that $E(B_{n,p}) = np$.