21-228 Week 6 notes

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1 More Recurrence Relations

Now let us look at second order linear recurrences. Consider the recurrence relations $a_{n+2} + 2a_{n+1} - a_n$.

Again. we apply a similar method:

$$\begin{aligned}
\sum_{n=0^\infty} a_{n+2}x^{n+2} &= 2\sum_{n=0}^{\infty} a_{n+1}x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} \\
\sum_{n=0}^{\infty} a_{n+2}x^{n+2} &= 2x\sum_{n=0}^{\infty} a_{n+1}x^{n+2} - x^2\sum_{n=0}^{\infty} a_n x^n \\
\sum_{n=2}^{\infty} a_n x^n &= 2x\sum_{n=1}^{\infty} a_n x^n - x^2\sum_{n=0}^{\infty} a_n x^n \\
\sum_{n=0}^{\infty} a_n x^n - a_1 x - a_0 &= 2x\left(\sum_{n=0}^{\infty} a_n x^n - a_0\right) - x^2\sum_{n=0}^{\infty} a_n x^n
\end{aligned}$$

Rearranging terms and placing the generating function on the left gives us:

$$\sum_{n=0}^{\infty} a_n x^n (1 - 2x + x^2) = (a_1 - 2a_0)x + a_0$$

Or, equivalently:

$$\sum_{n=0}^{\infty} a_n x^n = \frac{a_0 + (a_1 - 2a_0)x}{(1-x)^2}$$

As we have seen,

$$\begin{aligned}
1-x)^{-2} &= \sum_{k=0}^{\infty}\binom{2+k-1}{k}x^k \\
&= \sum_{k=0}^{\infty}\binom{k+1}{k}x^k \\
&= \sum_{k=0}^{\infty}(k+1)x^k
\end{aligned}$$

So by substitution we get:

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n &= (a_0+(a_1-2a_0)x)\sum_{k=0}^{\infty}(k+1)x^k \\
&= a_0\sum_{k=0}^{\infty}(k+1)x^k+(a_1-2a_0)\sum_{k=0}^{\infty}(k+1)x^{k+1} \\
&= a_0+a_0\sum_{k=1}^{\infty}(k+1)x^k+(a_1-2a_0)\sum_{k=1}^{\infty}kx^k \\
&= a_0+\sum_{k=1}^{\infty}\left(k\left(a_1-a_0\right)+a_0\right)x^k
\end{aligned}$$

We may extend this method to solve any recurrence of the following form:

$$a_{n+k}=\sum_{i=0}^{k-1} b_i a_{n+i}$$

1.1 Another Example

Now, consider the relation $a_{n+2}=a_{n+1}+2a_n$:

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2}=\sum_{n=0}^{\infty} a_{n+1}x^{n+2}+2\sum_{n=0}^{\infty} a_n x^{n+2}$$

So, moving all gen. functions to the left side we get:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)(1-x-2x^2)=a_1x+a_0-a_0x$$

So,

$$
\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n &= \frac{(a_1 - a_0)x + a_0}{1 - x - 2x^2} \\
&= \frac{a_0 + (a_1 - a_0)x}{(1-2x)(1+x)} \\
&= (a_0 + (a_1 - a_0)x) \left(\sum_{i=0}^{\infty} (2x)^i \right) \sum_{i=0}^{\infty} (-x)^j \\
&= (a_0 + (a_1 - a_0)x) \left(\sum_{i=0}^{\infty} 2^i x^i \right) \left(\sum_{j=0}^{\infty} (-1)^j x^j \right) \\
&= (a_0 + (a_1 - a_0)x) \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} s^i (-1)^{k-1} \right) x^k \\
&= (a_0 + (a_1 - a_0)x) \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=0}^{k} (-2)^i \right) x^k
\end{aligned}
$$

The formulafor the sum of a geometric series shows us that thecoefficient of x^k is $\frac{2^{k+1}}{3} + \frac{(-1)^k}{3}$. So, we see that

$$
\sum_{n=0}^{\infty} a_n x^n - \frac{a_0 + (a_1 - a_0)x}{3} \sum_{i=0}^{\infty} (2^{i+1} + (-1)^i) x^i
$$

It follows that:

$$
\begin{aligned}
a_i &= \frac{a_1 - a_0}{3} (2^i + (-1)^{i-1}) + \frac{a_0}{3} (2^{i+1} + (-1)^i) \\
&= \frac{a_1 + a_0}{3} 2^i + \frac{2a_0 - a_1}{3} (-1)^i
\end{aligned}
$$

Similar techniques are used to prove the following theorem. See the proof on Page 176 of Bogart.

Theorem 1.1. *If the polynomial $x^2 + bx + c$ has distinct roots r_1 and r_2, and the sequence $a_{n+2} + ba_{n+1} + ca_n = 0$, then there are constance c_1 and c_2 so that $a_n = c_1 r_1^n + c_2 r_2^n$.*

Bogart gives a much more detailed, but also much harder to memorize, statement. If you know that $a_n + c_1 r_1^n + c_2 r_2^n$, we can solve for c_1 and c_2 by setting $n = 1$ and $n = 2$ (or any distinct values, really) and solve the resulting linear system of equations.

2 Recurrence Relations and Derangements

In HW4, you used I/E to find the number of derangments with n people. Now, we will use generating functions to accomplish the same thing. This is a different kind of relation that what we have done in class.

We can count the number of derangements as follows. Let d_n be the number of derangements on an n-element set. There are $n!$ total bijections. For each k, there are $\binom{n}{k}$ ways to fix k elements, and the remaining $n-k$ must be "deranged".

Therefore, we have

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}$$

If we divide by $n!$, we get:

$$1 = \sum_{k=0}^{n} \frac{d_{n-k}}{k!(n-k)!}$$

So

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{d_{n-k}}{k!(n-k)!} \right) x^n \tag{1}$$

Let

$$d(x) = \sum_{m=0} \frac{d_m x^m}{m!}$$

This is often referred to as the *exponential* generating function relative to the sequence $(1, 1, 1, \dots)$.

The LHS of (1) is just $frac11 - x$. Let us investigate the right hand side.

Remember the product principle for generating functions:

$$\sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} b_n x^n) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (a_k x^k) b_{n-k} k x^{n-k} \right)$$

So let us split up the RHS of (1) as follows:

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \frac{d_{n-k}}{k!(n-k)!} \right) x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} x^k \frac{d_{n-k}}{(n-k)!} \right)$$

Thus, the RHS of (1) is the product of the generating functions of the sequences $a_n = \frac{1}{n!}$ and $\frac{d_m}{m!}$.

The generating function for a_n is $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$. The generating function for b_m is clearly $d(x)$ as we defined earlier.

It therefore follows that:

$$\frac{1}{1-x} = e^x d(x)$$

So:

$$d(x) = \frac{e^{-x}}{1-x}$$

Now, the generating function for e^{-x} is just $\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$.

So we have:

$$d(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^k}{k!} \right) x^n$$

So it follows that:

$$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

3 Intro to Probability

3.1 Examples

Suppose that we are performing some operation with randomness – for example, tossing a die or flipping a coin.

Definition 3.1. *Let Ω be a finite or countable (i.e. no bigger than the natural numbers) set. Ω will be referred to as the underlineprobability space. With Ω we associate a function $P : \Omega \to \mathbb{R}^+$, which determines the probability. $P(\omega)$ is referred to as the probability of ω. We require that $\sum_{\omega \in \Omega} P(\omega) = 1$.*

Examples:

1. Consider tossing a fair coin. Then $\Omega = \{H, T\}$, for Heads and Tails, and $P(H) = P(T) = 1/2$.
2. Single die: $\Omega = \{1, 2, \ldots, 6\}$, $P(\omega) = 1/6$ for each $\omega \in \Omega$.

Both are examples of a uniform distribution:

Definition 3.2 (Uniform Distribution). *If Ω is a finite probaility space, then P is a uniform distribution over Ω if $P(\omega) = \frac{1}{|\Omega|}$ for each $\omega \in \Omega$.*

3.1.1 Bernoulli Trials until Success

Suppose we are performing a test with two possible outcomes – perhaps, we have a "loaded" coin wher einstead of the usual probability $\frac{1}{2}$ for a head, the probability is p (which might equal $1/2$ anyway, or it might not.

The outcomes we are concerned about are the number of trials until heads occurs.

Then $\Omega = \mathbb{N}$. What is $P(n)$ for a given natural number n? The first success can be on the nth trial iff the first $n-1$ trials are all tails, and the nth is a head. What is the probability that the first $n-1$ trials are all tails? $(1-p)^{n-1}$. The probability that the last one is heads is p.

3.1.2 Rolling Two Dice

Suppose we have two dice. We can have different probability spaces depending on what information we wish to consider. The first possibility is where the value on each die matters (eg. backgammon) rather than the sum of the two values that come up.

Here, $\Omega = [6] \times [6]$, representing the value on each die as an ordered pair. Each outcome in this space has probability $\frac{1}{36}$ – so we have the uniform distribution. Thus, for each $\omega \in \Omega$, we have $P(\omega) = 1/36$.

Now, what happens if we only care about the sum of the values on the two dice?

Ω is $\{2, 3, 4, \ldots, 12\}$. Ten $P(\omega) = \frac{6-|7-\omega|}{36}$.

3.2 Events

Definition 3.3. *If Ω is a probability space, then A is an* <u>*event*</u> *if $A \subseteq \Omega$, and:*

$$P(A) = \sum_{\omega \in A} P(\omega)$$

Let us return to the example of two dice, where $\Omega = [6] \times [6]$.

Suppose that $A = \{x_1 + x_2 = 7\}$. What is $P(A)$? Well, we know that $A = \{(1,6), (2,5), \ldots, (6,1)\}$, so $|A| = 6$ and each element of A has probability $1/36$, so $P(A) = 1/6$.

3.2.1 Pennsylvania Lottery

Choose 7 numbers from 1 through 80. The state chooses 11, randomly. You win if all 7 of your numbers are among the state's 11. What are your chances of winning?

Well, the state can pick the 11 numbers in $\binom{80}{11}$ ways, so there are $\binom{11}{7}$ possible winning choices of 7 numbers. There are $\binom{80}{7}$ overall choices, so your chances of winning are

$$\frac{\binom{11}{7}}{\binom{80}{7}}$$

Another approach to this problem is to first consider that the state chooses its 11 numbers in $\binom{80}{11}$ ways. For one of these ways to win, it must pick all seven of yours, plus 4 of the remaining 73 – this can happen in $\binom{73}{4}$ ways. Thus your chances of winning are:

$$\frac{\binom{73}{4}}{\binom{80}{11}}$$

3.2.2 The Big Game or Powerball

This is a slightly different kind of lottery. Here, you choose 5 numbers out of 48 (or something like that) out of one big bowl of white numbers. Then, out of another bowl, you pick a gold-colored number out of 30.

You win if the five white numbers match the state's randomly chosen five white numbers, and if the gold-colored number also matches. What is your probability of winning?

The state can pick the white numbers in $\binom{48}{5}$ ways, and the gold number can be chosen in 30 ways. So there are $30\binom{48}{5}$ possible combinations that the state can pick from, and your combination must exactly match. So your chances of winning are:

$$\frac{1}{30\binom{48}{5}}$$

3.2.3 Poker

We pick 5 cards at random from a standard deck of 52 playing cards. There are thirteen values, from 2 to 10, and then J, K, Q, and A. Each value has four cards, one each of Spades, Hearts, Clubs, and Diamonds.

Exercises:

1. What is the probability that you will get four cards of the same value?
2. What is the probability that you will get three cards of one value and two of another value?
3. what is the probability that you will get three cards of one value, but *not* have one of the combinations in (1) and (2)

3.3 Birthday Paradox

Let us consider k people in a room. What is the probability that two of them have the same birthday (for simplicity, assume that no one is born on February 29).

Here, we let $\Omega = [365]^k$, with the uniform distribution. How many of these will have no two people with the same birthday?

There are $(365)_26 = \frac{365!}{339!}$ such possibilities, and 365^k total such arrangements. If $k = 26$, what does the probability work out to? With a calculator, we can see it's *less than* $1/2$ that no two have the same birthday!