21-228 Week 3 Notes

September 16, 2001

1 Multinomial Coefficients

Suppose that we have n objects, and for each i from 1 to m, k_i labels of type i. Suppose further that $k_1 + k_2 + \cdots + k_m = n$. Think that we have n chairs, m different colors, and we must paint k_i chairs using color i. How many ways can we do this?

We first may choose k_1 chairs to be color 1. Then there are $n - k_1$ chairs left, and we have colors 2 through m that we need to deal with. So the problem is reduced to first coloring chairs with color 1, then reducing the problem to coloring the chairs with colors 2 through m.

Definition 1.1. *Denote by $\binom{n}{k_1,k_2,\ldots,k_m}$ the number of ways to paint the colors of n chairs so that exactly k_i of the chairs are colored with color i. More formally, this is the number of ways to partition an n element set into a list of sets, where the ith set in the list has size k_i. In using this notation, we always require that $\sum_{i=1}^{m} k_i = n$.*

$$\binom{n}{k_1, k_2, \ldots, k_m} = \binom{n}{k_1}\binom{n-k_1}{k_2, \ldots, k_m}$$

So we have formed the essence of a recursive method of calculating the desired value. Now we need the base case. We saw before that the base case happens when $m = 2$, and we learned that:

$$\binom{n}{k_1, k_2} = \frac{n!}{k_1! k_2!}$$

This suggests a guess for a formula:

$$\binom{n}{k_1, \dots, k_m} = \frac{n!}{k_1! \dots k_m!}$$

And we can prove this using induction:

Theorem 1.2. $\binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{k_1! k_2! \dots k_m!}$

Proof. By induction. As seen before, this is true whenever $m = 2$, for all choices of n, k_1, and k_2 with $k_1 + k_2 = n$. suppose the statement is true for $m = m_0$, and consider the case $m = m_0 + 1$. From the above discussion, we know:

$$\binom{n}{k_1, k_2, \dots, k_{m_0} + 1} = \binom{n}{k_1}\binom{n - k_1}{k_2, \dots, k_{m_0} + 1}$$

Now, in the right hand side multinomial coefficient, we have m_0 terms on the bottom, so we use the induction hypothesis, as well as the formula for $\binom{n}{k_1}$ to get:

$$\begin{aligned} \binom{n}{k_1, k_2, \dots, k_{m_0} + 1} &= \frac{n!}{k_1!} \frac{n - k_1!}{k_2! \dots k_{m_0+1}!} \\ &= \frac{n!}{k_1! k_2! \dots k_{m_0+1}!} \end{aligned}$$

□

These are called *multinomial coefficients*, and they are a straightforward generalization of binomial coefficients. Indeed, we can just think of $\binom{n}{k} = \binom{n}{k, n-k}$. And as these binomial coefficients are useful in finding the coefficients of $(x+y)^n$ we can use multinomial coefficients to find the coefficients of $(x_1 + \cdots + x_m)^n$:

Theorem 1.3 (Multinomial Theorem). *$(x_1 + \cdots + x_m)^n$ is the sum of all possible $\binom{n}{k_1, \dots, k_m} x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}$, where $k_1 + \cdots + k_m = n$.*

Proof. Exercise. Runs along the lines as the proof of the binomial theorem. □

1.1 An Application: Lattice Paths

Let's say that we are in the plane, starting at $(0, 0)$. At each step we may go from (i, j) to $(i + 1, j)$ or $(i, j + 1)$. How many ways can I get from $(0, 0)$ to (x, y)?

There must be a total for $(x + y)$ steps taken, and I can pick any y of them to be going "up" in the plane. Therefore, there are $\binom{x+y}{y}$ total paths.

More generally, to get from (x_0, y_0) to (x_1, y_1) there are $\binom{x_1 - x_0 + y_1 - y_0}{y_1 - y_0}$ possible paths.

Let us return to the special case where $(x_0, y_0) = (0, 0)$. This time let us count the number of paths from $(0, 0)$ to (x, y) that do not ever cross the line $y = x$. For simplicity, let us assume that $x \leq y$ – this way, we can consider only the paths that are only *above* the line $y = x$. This is the same as the number of paths that never even *touch* the line $y = x - 1$. This can then be thought of as counting the *total* number of paths from $(0, 0)$ to (x, y), and subtracting from that the number of paths that touch the line $y = x - 1$. We can make our lives easier by shifting everything one by one notch, and considering the set of paths from $(0, 1)$ to $(x, y + 1)$ that touch the line $y = x$.

But we've still got to figure out how to compute that. Here's what we do: For each path that does touch the line $y = x$, we take the path the is before the *first* touch, and then rotate this portion about the line $y = x$. This results in a path from $(1, 0)$ to $(x, y + 1)$. Furthermore, every path from $(1, 0)$ to $(x, y + 1)$ *must* touch the line $y = x$, as $(1, 0)$ is below the line and $(x, y + 1)$ above it. Thus we have formed a bijection from the set of paths from $(0, 1)$ to $(x, y + 1)$ that touch the line $y = x - 1$, to the set of all paths from $(1, 0)$ to $(x, y + 1)$. (Why?) This latter number is just, as we have seen, $\binom{x-1+y+1}{y+1}$, so the total answer is $\binom{x+y}{y} - \binom{x-1+y+1}{y+1}$. You can read the book to see the expansion of that – but in the end you get

$$\binom{x+y}{x}\frac{x-y+1}{x+1}$$

If $x = y = n$, we get $\frac{1}{n+1}\binom{2n}{n}$.

This is called the *Catalan Number*

2 Equivalence Relations

Equivalence relations are a specific kind of relation largely used to classify objects. Two objects that are not necessarily the same, may still share common properties that make them, for certain purposes, equivalent.

This motivates the idea of what should happen if we have a notion of equivalence. Clearly, any object should be equivalent to itself. Furthermore, if A is equivalent to B, we also would like B to be equivalent with A – we don't want to worry about which one comes first. Finally, if A is equivalent to B, and B is equivalent to C, this indicates that A shares some properties with B, and that B shares these properties with C – so we would like to also have that A is equivalent with C.

Note that equivalence only deals with pairs of elements – just like you can't have "$a =$" without anything on the right hand side. This suggests the idea of combining equivalence and relations, and having the concept of *equivalence relations*: For an arbitrary relation R, we often use the notation aRb to mean $(a, b) \in R$

Definition 2.1 (Equivalence Relation). *A relation R on a set A is an equivalence relation on A if:*

1. *For each $a \in A$, aRa.*
2. *For each $a_1, a_2 \in A$, If $a_1 R a_2$ then $a_2 R a_1$.*
3. *For each $a_1, a_2, a_3 \in A$, if $a_1 R a_2$ and $a_2 R a_3$, then $a_1 R a_3$.*

Example: Let $\mathbb{N}$ be the set of natural numbers. Let n be any given natural number. If a, b are integers, we say that $a|b$ if there is an integer c so that $ac = b$. Let us say that xRy iff $n|(x-y)$. Then R is an equivalence relation on $\mathbb{N}$.

1. aRa because $n|0 = (a-a) - n \cdot 0 = 0$.
2. If aRb then $n|(a-b)$, so there is a c so that $nc = (a-b)$. So $n(-c) = (b-a)$, so $n|(b-a)$, so bRa.
3. If aRb and bRc, then $n|(a-b)$ and $n|(b-c)$. So there is a d so that $nd = (a-b)$ and an e so that $ne = (b-c)$. Therefore, $nd + ne = (a-b) + (b-c) = a - c$, so $n(d+e) = a - c$, so $n|(a-c)$, so aRc.

We can also, given an element a, think about all of the objects in A that are equivalent to a. We call this the *equivalence class* containing A"

Definition 2.2 (Equivalence Class). *Let R be an equivalence relation on a, let $a \in A$. Then $\{b \in A \mid aRb\}$ is called the equivalence class of a under R.*

Fact 2.3. *If R is an equivalence relation on A, and aRb, then a and b have the same equivalence class under R.*

Proof. Exercise. □

We now define another concept that is closely related to that of an equivalence relation:

Definition 2.4 (Partition). *Let A be a set. A partition of A is a set P of subsets of A so that:*

1. *Each $S \in P$ is nonempty*
2. *Each element of a is in some element S of P.*
3. *If $S_1, S_2 \in P$, and $S_1 \neq S_2$, then S_1 and S_2 are disjoint.*

And now the way in which these are related:

Theorem 2.5. *Let R be an equivalence relation on A. Let C be the set of equivalence classes of R (remember, two equivalence classes that have the same elements count as the same calss). Then C is a partition of A*

Proof. Note that any equivalence class is necessarily nonempty. Furthermore, each $x \in A$ is in some equivalence class. If C_1 and C_2 are equivalence classes, and $c \in C_1 \cap C_2$, then we know that both the elements in C_1 and the elements in C_2 all relate to c. But that means that by Fact 2.3, all of the elements in $C_1 \cup C_2$ must have the same equivalence class, thus meaning that $C_1 = C_2$. Thus, if $C_1 \neq C_2$, then C_1 and C_2 must be disjoint. □

Theorem 2.6. *If P is a partion of a set X, then there is one and only one equivalence relation whose equivalence classes are the classes of P.*

Proof. It is fairly clear that if we define the equivalence classes of any relation, then we also define the relation itself – for each $x \in X$, we know exactly for which y we have xRy. Furthermore, the relation formed in this way is an equivalence relation, by our previous results. The details are left as an exercise (see #7 in section 2.1 of Bogart). □

2.1 Counting Partitions

Counting the number of equivalence relations on a set A is a very diffcult task. Later on, we will introduce more powerful counting techniques to help us perform this task. For now, let us simplify things. Let us suppose that we have a group of n people. We need to split this group into k_1 groups of size 1, k_2 groups of size 2, and so forth, up to k_n groups of size on (clearly, k_n can never be more than 1!). Then, assuming that $\sum_{i=1}^{n} ik_i = n$, we have:

Theorem 2.7. *Let A be a set of size n. If $k_1, \ldots, k_n$ are so that $\sum_{i=1}^{n} ik_i = n$, then there the number of partitions of A so that k_i elements of the partition have size i is:*

$$\frac{n!}{\prod_{i=1}^{n} ({}^{j_i} \cdot j_i!)}$$

Proof. We can think about listing the elements of A in order. The first j_i elements listed will be those in a set of size 1, the next $2j_2$ elements will be those in a set of size 2, and so forth. Let us now focus our attention on classes of size k. When we list these elements, the first k of these will be in the first set, the next k will be in the second set of size k, and so forth. Now, within each set, there are $k!$ ways of listing elements in each set – furthermore, there are $j_k!$ ways of rearrangling the sets themselves – since the order of the sets, or the order of the elements in each set does not matter, the by the product principle, we must divide by $(k!)^{j_k}$, and again by $j_k!$. We therefore get:

$$\frac{n!}{\prod_{i=1}^{n} ({}^{j_i} \cdot j_i!)}$$

□

3 Distributions and Multisets

We may think of the counting problems we have given so far as ways of distributing "labels" among a group of recipients. We will continue in this section to try and solve more problems of this type. Examples that we have solved so far:

1. Distributing k distinct labels to a set of n distinct recipients – no restrictions – if a recipient gets more than one label, the order in which the labels are placed is irrelevant.

2. Distributing k distinct labels to a set of n distinct recipients – each recipient gets at most 1.

3. Distributing k identical labels to a set of n distinct recipients – each recipient gets at most 1.

(I've spent loads of time attempting to parse the table on P.85 of Bogart and have still failed miserably).

Let's try a few more. Suppose we have k books and a bookshelf with n shelves. We care now both about which shelf each book goes on as well as the order of the books on a given shelf. (Thus, the answer will be somewhat greater than the n^k that would occur if we did not care about the order of books on a given shelf).

Here is how we solve the problem. Suppose that we have already placed m books, and we wish to place the $m+1$st book. Note that as we place more books, the number of possible "different places" that each book my go will increase. Say that there were k_m possible places for the mth book to go. Once we placed the mth book, we could in essence have placed the $m+1$st book into the same number of places. If we placed it in the spot that m had ended up in, then in fact we would place it either before or after book m – thus the spot that book m occupies has really been split up into two available spots for book $m+1$. Therefore, there are k_m+1 possible places for book $m+1$ to go.

Hence, the first of the k books may go into each shelf – or any of n places, the next may go into $n+1$ places, the third may go into $n+2$. In general, we inductively see that the mth book may go into $n+m-1$ places.

Therefore, we see that we have $\prod_{i=1}^{k}(n+i-1)$ total such arrangements.

If instead we stipulate that the books are all identical instead of distinct, we divide the above by $k!$, which works out to $\binom{n+k-1}{k}$, since once we have distributed them, any "swapping" around results in distributions that are no longer different.

We can also think of this last problem in terms of giving identical pieces of candy to children. What if we want to ensure that each child gets at least one piece? We first set aside n pieces for each child, then distribute the remainder as above.

4 Multisets

One final application is we can ask what happens if we have a situation where we want to choose elements from a set, without regard for order, but now we are allowed to choose an element more than once.

Example: How many fruit baskets with 10 pieces of fruit can be constructed using any number of oranges, apples, pears, and bananas?

This can be solved in the same way as giving candy to the children. Why?

Theorem 4.1. *Thus, if we wish to select k objects from a group of n, with repetition allowed, where order of selection of objects does not matter, we have a total of*

$$\binom{n+k-1}{k}$$

possibilities.

4.1 Broken Permutations

Let's suppose we have a set of k elements, and we want to break it into n different stacks, so that each element appears in some stack, and where the order of the stacks matters. We also require that each stack have at least one element.

Then we take the ordered distributions, and by first picking the first element in each stack, we get:

$$\frac{k!}{(k-1)!}(k-n)!(n-1)! = k!\binom{k-1}{n-1}$$

Such distributions. But we also need now to divide by $n!$ to account for the fact that the recipients are identical.

4.2 Problems We have Solved

So, now we have the following list of questions and their answers:

1. If we have k distinct books and n distinct shelves, there are $\prod_{i=1}^{k} n-i+1 = \frac{(n+k-1)!}{(n-1)!}$ arrangements.
2. If we have k identical books and n distinct shelves, then there are $\binom{n+k-1}{n-1}$ arrangements.
3. If we have k distinct books and n identical shelves, then we have $\frac{(n+k-1!)}{(n-1)!n!}$ arrangements.
4. If we have k distinct books and n distinct shelves, and we require that each shelf get at least one book, then we have $\frac{(k-1)!}{(n-1)!}$ arrangements.
5. If we have k identical books and n distinct shelves, and we require that each shelf get at least one book, then we have $\binom{k-1}{n-1} = \binom{k-1}{k-n}$ arrangements.
6. If we have k distinct books and n distinct shelves, and we require that each shelf get at least one book, then we have $\binom{k}{n}\frac{(k-1)!}{(n-1)!}$ arrangements.

We still do not know exactly what to do if both the books and the shelves are identical. We will try and work towards solving this problem soon.

5 Mathematical Games

Here, we get into a discussion of mathematical games, and how we may solve them.

5.0.1 Take-Away

Suppose we have a stack of 17 chips. Two players play, and each player in his or her turn may remove up to 4 chips in a turn. The player to remove the last chip wins. What is a winning strategy for the first player?

We solve problems like this by working backwards. The player to have left zero chips wins. So we want to move with 1, 2, 3, or 4 chips remaining. If there are 5 chips remaining, we see that no matter what we do, we must leave our opponent with 1, 2, 3, or 4 chips. So we want to force our opponent to move with 5 chips remaining. To this end, we want to ourselves move with 6-9 chips remaining. We can do this my making our opponent move with 10 chips remaining.

Definition 5.1. *A game in a state where the player that is about to move can force a win with optimal play is called an N-position. A game in a state where the player that has just moved can force a win is called a P-position.*

So, if we're moving, we want to be in an N-position, and want to force the game into a P-position.

Note that if we are in a game where all moves lead to an N-position, then we are in a losing position (i.e. a P-position). Also, if there is even one move that leads to a P-position, then the player about to move can win.

In terms of the game that we have just described, a position where there are no chips remaining is a P-position, and if there are 1-4 chips remaining, we are in an N-position. So, since all moves from a position with 5 chips remaining lead to N-positions, then the position with 5 chips remaining must be a P-position. Further, any position with 6-9 chips remaining has a move to a P-position (namely, the one with 5 chips remaining), these positions are N-positions. And so on and so forth, we see that any position where there are a multiple of 5 chips remaining is a P-position, and all others are N-positions.

If we are playing a game so that

- The game must terminate, and there can be no draws.
- There is no randomness and all information in the game is available to all players (unlike in cards or backgammon).
- Players must alternate moves – no simultaneous moves are allowed (eliminating paper-rock-scissors)

Then it turns out that any position in this game must have a guaranteed win for one side or the other.

In terms of take-away, this means that no matter how many chips are in the pile, *someone* must be guaranteed a win.

5.1 Strategy-Stealing

Consider a generalized version of tic-tac-toe. Here, we allow ourselves to have an $m \times n$ board, and instead of winning whenever we have 3 in a row, we require that we have k in a row, for some fixed k.

We can prove that the second player to move in such a game can never have a winning strategy. Now, since tic-tac-toe can have draws, this does not imply that the first-player has a winning strategy (indeed, the familiar tic-tac-toe, where $m = n = k = 3$ of course has no winning strategy for either player).

In tic-tac-toe, we associate a position not only the contents of the board, but also the player whose turn it is.

Then, we have:

Fact 5.2. *In a position of tic-tac-toe, if a position is winning for a player then that position, minus one piece of an opposing player's type, must also win for that player. Further, adding one piece of the same player's type also yields a winning position for that player.*

In other words, having "extra" pieces is *always* good (unlike chess or checkers, where there are cases where extra pieces may backfire on us).

Now, suppose that the second player to move has a winning strategy – in other words, that an empty board is a P-position.

Suppose the first player is "X" and the second player is "O". After "X" moves, then "O" moves to place the game into a position where "X" can move, but the game is still a guaranteed win for "O". But if we then swap the "X"s and "O"s, and now say that "O" has the next move, this means that now the game is a guaranteed win for "X". If we then remove the "O" that is on the board, we are at a position where "X" could have moved originally, that is a guaranteed win for "X", even if it is "O"'s move! This means that "X" must have a winning strategy too. But both players can't have a winning strategy. Thus the assumption that "O" had a winning strategy is a contradiction.