1 Matchings

Before, we defined a *matching* as a set of edges no two of which share an end in common. Suppose that we have a set of jobs and people and we want to match as many jobs to people as we can.

So we are given a bipartite graph with vertex set $V = A \cup B$ and edge set E, where A is the set of people and B is the set of jobs – we want to match up as many jobs as possible.

Another application may be if we have a graph where V is just the set of people, and two people are adjacent iff they are compatible to be roommates – we wish to make as many roommate pairs as possible.

One attempt may be to just blindly pick edges, so that no two edges share a vertex in common, until we cannot do anymore. A matching obtained in this manner is called *maximal*.

Definition 1.1. *A matching M in a graph G is called maximal if there is no matching M' in G so that $M \subsetneq M'$.*

Definition 1.2. *A matching M in a graph G is maximum if there is no matching M' in G so that $|M| < |M'|$.*

It is obvious that all maximum matchings are also maximal. But the converse is not true. Can you come up with an example of a graph G with a maximal matching that is not maximum?

eg. A path of odd length. The even-numbered edges in the path are a maximal matching, but not maximum.

So, the problem of finding a maximum matching is not so simple as picking edges greedily. We need a way to figure out if a matching is maximum or not.

Definition 1.3. *Let $G = (V, E)$ be a graph, and M a matching in G.*

1. *If v is the end of some edge in M, we say that M saturates v. Otherwise, we say that v is unsaturated.*

2. *An M-alternating path is a path whose edges alternate between edges in M and edges not in M.*

3. *An M-augmenting path is an M-alternating path whose ends are both not saturated by v.*

If a graph has an M-augmenting path, then M cannot be a maximum matching. This is not so hard to see. What is harder to see is that the converse is also true.

Theorem 1.4. *M is a maximum matching in G if G has no M-augmenting path.*

Proof. Suppose M is not maximum. Let M' be a maximum matching in G, and consider the set $M \triangle M'$ consisting of $M \cup M' - (M \cap M')$, and consider the graph G^* consisting only of these edges. Since M and M' are matchings, each vertex has at most one incident edge from each of them, and this subgraph has maximum degree at most 2. So G^* consists of disjoint paths and cycles, and each path and cycle must alternate between edges in M adn edges in M'. Some component of this graph must have more edges from M' than from M, and this must be an M-augmenting path. $\square$

1.1 Duality

So, we can check whether M is maximum by tring to find an M-augmenting path. However, if M is maximum, it is not easy to prove that no such path exists. There is another concept, known as duality.

The idea of duality runs as follows – we have two optimization problems. We wish to find the biggest matching, in this case. What we would like to also have is a related minimization problem – the idea is that any answer to the minimization problem should have a larger answer than any answer to the maximization problem.

In this case, suppose we find a matching of size 7. Then all answers to our corresponding minimization problem should always have size greater than seven. Furthermore, what we would like to have happen is that if we find a maximum matching (in this case), we should find a minimum (something) hat has the same "value" as the maximum answer.

The optimization problem we use is as follows:

Definition 1.5. *A vertex cover in a grpah G is a set of U vertices in G so that every edge has at least one end in U.*

I claim that any vertex cover of a graph is as big as any matching.

Theorem 1.6. *Let G be a graph, M be a matching in G and U be any vertex cover in G. Then $|U| \geq |M|$.*

Proof. Let M be a matching. Any vertex cover in G must use one vertex from each edge in M – since these ends are all distinct, there must be at least as many vertices in U as there are edges in M. $\square$

Corollary 1.7. *If M is a matching in G and U is a vertex cover in G, and $|M| = |U|$, then we know that M is a maximum matching and U is a minimum vertex cover.*

The nice thing about duality is that if we want to prove something (eg. a matching) is maximum, we now have a way of doing so without having to check all the matchings. We construct a vertex cover of the same size, and the very fact that we were able to do so gives a *constructive* and easy-to-check proof that M is a maximum matching. Duality has many very important applications in optimization.

It turns out, that in bipartite graphs, if M is a maximum matching, we can *always* find a vertex cover U of the same size as M. We will not prove this result here.

2 Planarity

For the remainder of this course, assume all graphs are simple.

Recall first the definition of Isomorphism

Definition 2.1. *Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic iff there is a bijection $f : V_1 \to V_2$ so that $f(u)$ and $f(v)$ are adjacent in G_2 whenever u and v are adjacent in G_1.*

So, two graphs are isomorphic if one can be made into the other by just relabeling vertices and redrawing the edges (without allowing them to change ends).

One topic of interest is asking if we can draw a graph on a piece of paper so that the edges do not cross.

For instance, we can do so for the complege graph on 4 vertices (each pair of vertices has an edge).

Definition 2.2. *A graph $G = (V_1, E_1)$ is called planar if it can be drawn in the plane with no two edges crossing.*

We would like a simple rule for determining whether a graph is planar:

Theorem 2.3. *For any drawing of a graph on the plane, $v + e - f = 2$, where v is the number of vertices, e is the number of edges, and f is the number of regions into which the plane is split.*

Proof. See Bogart, p.246 □

To prove a graph is planar, we just have to exhibit a drawing in the plane (these are called *planar embeddings*) for which the edges do not cross.

Theorem 2.4. *In a connected planar graph with at least three vertices, $e \leq 3v - 6$.*

Proof. Each face has three edges or more, and each edge either separates (or does not separate) two faces. Consider the set of all ordered pairs $(edge, face)$ where the edge touches the face. There are at least $3f$ such pairs, but at most $2e$. So $3f \leq 2e$, and $f = 2 + e - v$, so we get $6 + 3e - 3v \leq 2e$, whence $e \leq 3v - 6$. □

Corollary 2.5. *K_5, the complete graph on 5 vertices, is not planar.*

Proof. $v = 5$, and $e = 10 > 9 = 3 \cdot 5 - 6$. □

Similarly,

Corollary 2.6. *$K_{3,3}$, the complete bipartite graph with 3 vertices on each side, is not planar.*

Proof. Since a bipartite graph contains no odd cycles, each face in a planar embedding must have *four* sides. Therefore, we get $4f \leq 2e$. Therefore, we see that $f \leq 4.5$. But if we have 6 vertices and 9 edges, we must have 5 faces. □

Definition 2.7. *A* *subdivision* *of a graph* G *is a graph* G^* *obtained by placing a vertex in the middle of an edge.*

Theorem 2.8. *A graph is planar iff any subdivision of the graph is planar.*

Proof. Exercise. □

Theorem 2.9. *A graph is planar iff it contains no subdivision of* $K_{3,3}$ *or* K_5*.*

3 Directed Graphs

We have already basically touched on directed graphs. Remember that the definition is modified so that each edge has an associated direction.

3.1 Euler Paths in Directed Graphs

Remember we had a theorem about Euler Tours/Paths in undirected graphs. We have a similar theorem about Euler Torus in directed graphs.

We denote an edge by uv as usual, but in a directed graph all "motion" must go *from* u *to* v. u is called the tail and v the head.

Definition 3.1. *If* D *is a directed graph, the indegree of a vertex* v *is the number of edges* uv *with head at* v *and the outdegree is the number of edges* vu *with talk at* v*. The degree of* v *is the sum of the indegree and outdegree of* v*.*

Theorem 3.2. *Let* $D = (V, E)$ *be a directed graph.*

1. *D has an Euler Tour iff it is connected and the indegree of each vertex equals the outdegree of each vertex.*
2. *D has a nonclosed Euler walk iff it is connected and all but two vertices have equal indegree and outdegree equal. Of the remaining two, one should have indegree one more than outdegree, and the other should have outdegree one more than indegree.*

The proof is similar to that for undirected graphs (in fact it's a bit easier in this case).

3.2 Tournament Digraphs

Definition 3.3. *If* $G = (V, E)$ *is an undirected graph, then a digrap* D *with vertex* V *is called an orientation of* G *if* D *can be obtained by assigning a head and tail to each edge of* G*.*

Definition 3.4. *A* *tournament* *is an orientation of a complete graph.*

Theorem 3.5. *Every tournament has a directed hamiltonian path*

Proof. Let v_1 be any vertex so that v_1v_2 is an edges. We thus have two vertices and a path joining them Suppose now we have a path $(v_1, (v_1v_2), v_2, \dots (v_{i-1}, v_i)v_i)$ of length $i-1$ Suppose that v_0 is not on this path. If (v_0, v_1) is an edge, we may place v_0v_1 at the beginning. If v_iv_0 is an edge me may do similarly. Otherwise, consider the smallest k (which must be at least 2) so that v_kv_0 is not an edge. Then v_0v_k is an edge and so is $v_{k-1}v_0$. We can then splice $v_{k-1}v_0, v_0, v_0v_k, v_k$ into the path to get a longer one, and continue until we have used all vertices. □

Definition 3.6. *A directed cycle is a closed directed walk that does not use any edge or vertex more than once (except the start vertex).*

Definition 3.7. *A digraph is transitive if whenever (x, y) and (y, z) are edges then so is (x, z).*

Theorem 3.8. *A torunament without a directed cycle is transitive.*

Proof. Exercise. □

4 coloring

Let us now investigate coloring vertices of a graph. A "proper" coloring of the vertices of a graph is one where no two adjacent vertices have the same color. We want to find the smallest such covering possible.

The smallest such number of colors we need to properly color G is called the *vertex-chromatic* number of G (we can define the *edge-chromatic* number similarly, and this also is an interesting problem). We will also call a graph k-vertex-colorable (or edge if appropriate) if it can be "properly" colored with k colors.

For this section, all colorings are assumed to be vertex colorings.

Under what conditions is a graph 2-colorable?

Colorability on more than three colors is much more challenging. This problem is what we call NP-complete. There are a large class of NP-complete problems, and if we know how to solve one, then we know how to solve them all.

Theorem 4.1. *If the largest degree of a vertex in G is k, then we can color G using at most $k + 1$ colors.*

Proof. Color "greedily" picking whichever color we feel like at each step as long as we don't pick the same color as a neighbor. Why does this work? Each vertex has at most k neighbors, so there is always one free color at each step. □