1 Paths and Proofs

Now that we have looked at the first example of a "classic" Graph Theoretic problem, we return to more "basics"

We have already defined walks, closed walks, and trails. We now add on to this:

Definition 1.1. *Let $G = (V, E)$ be a graph*

1. *A path in G is a walk of length at least one that does not use any vertex more than once.*
2. *A cycle in G is a closed walk of length$(v_0, e_1, v_1, \ldots, e_k, v_k)$ of length at least one, so that the only pair of vertices that are the same is (v_0, v_k).*

We have gone over connected graphs and vertices. We also defined connectivity of two vertices u, v. It is not hard to see that connectivity is an equivalence relation.

Definition 1.2 (Induced Subgraph). *If $G = (V, E)$ is a graph, and $U \subseteq B$, the subgraph $G[U]$ with vertex set U and whose edges have both ends in U is called the subgraph of G induced by U.*

Definition 1.3. *Let $G = (V, E)$ be a graph. The subgraphs induced by the equivalence classes of the "connectivity" relation on V are called components.*

Fact 1.4. *If u and v are distinct vertices in G, then every u, v-walk (i.e. a walk from u to v) in G contains a u, v-path in G.*

Proof. Suppose that G is a walk W that has a repeated vertex. That is, $W = (v_0, e_1, v_1, \ldots, e_k, v_k)$, where for some distinct i, j, $v_i = v_j$. Then we may remove the segment $e_{i+1}, v_{i+1}, \ldots, e_j, v_j$ to have another perfectly valid walk, as e_{j+1} must leave $v_j = v_i$. We may repeat this process indefinitely. The walk will either decrease in length by at least one each time, until there is no repeated vertex left. Therefore, after a finite number of iterations, we will have a path. This proof can be phrased more concisely using strong induction. □

1.1 Cut-Edges and Cut-vertices

Definition 1.5. *Let $G = (V, E)$ be a graph. Let $G - \{v\}$ be the graph resulting by removing vertex v and all edges incident with v. Let $G - \{e\}$ be the graph resulting by removing edge e. If $G - \{v\}$ or $G - \{e\}$ has more components than G, we call v or e a cut-vertex or cut-edge, respectively.*

Fact 1.6. *A graph $G = (V, E)$ is connected iff for every partition of $V = A \cup B$, there is an edge E with one end in A and the other in B*

Proof. Suppose that G is connected. Let A, B be a partition of V. Then there must be an edge with one end in A and the other in B, for otherwise there would be no walk from $a \in A$ to $b \in B$. (any such walk $(v_0, e_1, v_1, \ldots, e_k, v_k)$

must have some subsequence (v_i, e_{i+1}, v_{i+1}), where $v_i \in A$ and $v_{i+1} \in B$, but there can be no appropriate e_{i+1}.

Now, suppose that G is disconnected. Then there are two vertices $u, v \in V$ that are not connected. Thus, u and v are in two different components C_1 and C_2 of G. The vertices $V(C_1)$, and $V \setminus V(C_1)$ therefore partition V and there is no edge from $V(C_1)$ to $V \setminus V(C_1)$ – else they would not be two different sets. □

Theorem 1.7. *An edge of a graph is a cut-edge iff it belongs to no cycle*

Proof. Suppose first that an edge e of $G = (V, E)$ belongs to a cycle. The only way in which the number of components could increase is if the component C containing the ends of e were to become disconnected. So let u, v be in $V(C)$. We wish to show that there must still be a path from u to v, without using e. Let W be any arbitrary walk from u to v. If W already does not use e then we are done. Else, suppose that W uses $e = ab$. Then ab are adjacent edges of a cycle, and $W = (v_0, e_1, \ldots, a, e, b, \ldots, e_k, v_k)$. Then we may replace e with the path around the cycle in the other way (draw a picture to represent this). Thus, there is still a walk from u to v.

Now, suppose that $e = ab$'s removal does not increase the number of components. Then there is a path from a to b that does not use e. Adding the edge e to the path after b results in a cycle. □

2 Bipartite Graphs

Definition 2.1. *A graph G is bipartite if $V(G)$ can be partitioned into two sets A, B so that every edge has one end in A and another in B.*

Fact 2.2. *A graph is bipartite iff it has no cycle of odd length*

Proof. Suppose first that a graph G has a cycle of odd length. Then one can see that even partitioning these vertices must necessarily violate the needed requirements.

For the other direction, use a greedy algorithm. □