21-228 Homework 5 Solutions

October 20, 2001

1. Use generating functions to solve $a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n$ **where** $a_0 = a_1 = a_2 = 1$

The sequence is, trivially, $a_n = 1$ for all n.
We have:

$$\sum_{n=0}^{\infty} a_{n+3}x^{n+3} = 3\sum_{n=0}^{\infty} a_{n+2}x^{n+3} + 3\sum_{n=0}^{\infty} a_{n+1}x^{n+3} + \sum_{n=0}^{\infty} a_n x^{n+3}$$

Therefore:

$$\sum_{n=0}^{\infty} a_n x^n - (a_2x^2 + a_1x + a_0)$$
$$= 3x\left(\left(\sum_{n=0}^{\infty} a_n x^n\right) - (a_1x + a_0)\right) - 3x^2\left(\left(\sum_{n=0}^{\infty} a_n x^n\right) - a_0\right) + x^3\sum_{n=0}^{\infty} a_n x^n \tag{1}$$

Lettng $S = \sum_{n=0}^{\infty} a_n x^n$, we get:

$$S - (a_2x^2 + a_1x + a_0) = 3x(S - (a_1x + a_0)) - 3x^2(S - a_0) + x^3$$

We need now to solve for S:

$$(1 - 3x + 3x^2 - x^3)S = -3x(a_1x + a_0) + 3x^2a_0 + a_2x^2 + a_1x + a_0$$

Now, substituting $a_2 = a_1 = a_0 = 1$ we get

$$(1-x)^3 S = -3x(x+1) + 3x^2 + x^2 + x + 1$$

So

$$S = \frac{x^2 - 2x + 1}{(1-x)^3} = \frac{1}{1-x}$$

So, $S = \sum_{n=0}^{\infty} x^n$, and it follows that $a_n = 1$ for all n.

2. Solve the following variant of Fibonacci's problem. Each mature pair of rabits present at the end of any given month produces three more pairs of rabbits during the *next* month. Further, the baby rabbits take a month to mature. If we start with 10 baby rabbits (so no procreation takes place until the second month), how many rabbits are present after n months?

Let a_n be the number of rabbits present at the end of the nth month. a_0 is the number of rabbits at the *beginning* of the 1st month (why?). We thus have $a_0 = a_1 = 10$ to start the recurrence.

Then $a_{n+2} = a_{n+1} + 3a_n$.

So we have $a_{n+2} - a_{n+1} - 3a_n$.

So we wish to solve the polynomial $r^2 - r - 3 = 0$. The roots of this polynomial are $\frac{1 \pm \sqrt{13}}{2}$.

Therefore, we have:

$$a_n = c_1 \left(\frac{1+\sqrt{13}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{13}}{2} \right)^n$$

For some constants c_1 and c_2.

Since $a_0 = 10$, we have:

$$10 = c_1 + c_2 \tag{2}$$

And since $a_1 = 10$ we have

$$10 = \frac{c_1}{2} + \frac{\sqrt{13}}{2} c_1 + \frac{c_2}{2} - \frac{\sqrt{13}}{2} c_2$$

Since $c_1 + c_2 = 10$, we get

$$10 = \sqrt{13}c_1 - \sqrt{13}c_2 \tag{3}$$

Multiplying (2) by $\sqrt{13}$ we get

$$10\sqrt{13} = \sqrt{13}c_1 + \sqrt{13}c_2 \tag{4}$$

So

$$\begin{aligned} c_1 &= \frac{10(\sqrt{13}+1)}{2\sqrt{13}} \\ &= 5 + \frac{5\sqrt{13}}{13} \end{aligned}$$

And

$$c_2 = 5 - \frac{5\sqrt{13}}{13}$$

So

$$a_n = (5 + \frac{5\sqrt{13}}{13})\left(\frac{1+\sqrt{13}}{2}\right)^n + (5 - \frac{5\sqrt{13}}{13})\left(\frac{1-\sqrt{13}}{2}\right)^n$$

3. Suppose that we have the recurrence relation $a_{n+2} - 2ra_{n+1} + r^2a_n = 0$, for some real nonzero r. Find a general solution for the relation in terms of a_0 and a_1.

We have

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 2r\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + r^2\sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Therefore:

$$\sum_{n=0}^{\infty} a_n x^n - 2rx\left(\sum_{n=0}^{\infty} a_n x^n - a_0\right) + r^2x^2 \sum_{n=0}^{\infty} a_n x^n = a_1 x + a_0$$

So

$$\sum_{n=0}^{\infty} a_n x^n - 2rx \sum_{n=0}^{\infty} a_n x^n + r^2x^2 \sum_{n=0}^{\infty} a_n x^n = (a_1 - 2ra_0)x + a_0$$

So:

$$(1 - 2rx + r^2x^2) \sum_{n=0}^{\infty} a_n x^n = (a_1 - 2ra_0 x) + a_0$$

And:

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n &= \frac{a_1 x - 2ra_0 x + a_0}{(1-rx)^2} \\ &= (a_1 - 2ra_0)x\frac{1}{(1-rx)^2} + \frac{a_0}{(1-rx)^2} \end{aligned}$$

We know that $(1 - rx)^{-2} = \sum_{n=0}^{\infty} nr^n x^n$.
So we get

$$\sum_{n=0}^{\infty} a_n x^n = (a_1 - 2ra_0) \sum_{n=1}^{\infty} (n-1) r^{n-1} x^n + a_0 \sum_{n=0}^{\infty} nr^n x^n$$

So

$$a_n = (a_1 - 2ra_0)(n-1)r^{n-1} + a_0 nr^n$$

4. Solve the recurrence relation $a_{k+1} = a_k + 2^k$ with $a_0 = 2$.

This can be done using generating functions, but there is also a simple inductive proof.

Claim: $a_n = 2^{n+1}$ for all n

Proof: By induction.

Base Case: The statement is clearly true for $n = 0$.

Inductive Step: Suppose now that $a_k = 2^{k+1}$. Then $a_{k+1} = 2^{k+1} + 2^{k+1} = 2^{k+2}$.