21-228 Exam 2

Name:____________________________________

November 7, 2001

- Place your name and Section letter (whichever section you are *sitting in on*) on the space provided.
- You have 50 minutes. Pace yourself appropriately.
- **All answers must be justified to receive credit**. Please write as legibly as possible.
- You may use the back sides (and any other spare space on the exam pages) if you wish. Clearly indicate where your work is if you wish to do so – we will not be responsible for grading such work otherwise.
- The back page contains a formula sheet, which you may tear off if desired.
- Good luck!

Question	Score	Possible
1		20
2		20
3		20
4		20
5		20
Total		100

1. (20 points) Use generating functions to solve:

$$a_{n+1} = 3a_n + 2^n, a_0 = 0$$

Multiplying both sides by x^{n+1} and summing from $n = 0$ to infinity gives:

$$\sum_{n=0}^{\infty} a_{n+1}x^{n+1} = \sum_{n=0}^{\infty} 3a_n x^{n+1} + \sum_{n=0}^{\infty} 2^n x^{n+1}$$

Let $S = \sum_{n=0}^{\infty} a_{n+1}x^n$. Then we have:

$$S - a_0 = 3xS + \sum_{n=0}^{\infty} 2^n x^{n+1}$$

Since $a_0 = 0$, the above simplifies to

$$S - 3xS = \sum_{n=0}^{\infty} 2^n x^{n+1}$$

Dividing both sides by $(1 - 3x)$ gives:

$$S = \frac{\sum_{n=0}^{\infty} 2^n x^{n+1}}{1 - 3x}$$

Factoring x out of the sum on the right hand side, and using the formula for geometric sum on the right hand side gives

$$S = \frac{x}{(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x}$$

It now remains to solve for A and B. We know that $A(1-3x) + B(1-2x) = x$ for all x. So, if $x = 0$ this gives $A + B = 0$. If $X = 1$ we get $-2A - B = 1$. Adding these two equations gives $-A = 1$, whence $B = 1$

So

$$S = \frac{1}{1-3x} - \frac{1}{1-2x} = \sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} 2^n x^n$$

So $a_n = 3^n - 2^n$.

2. (20 points) Solve the following recurrence relations:

(a) (10 points) $a_{n+2} = 5a_{n+1} - 6a_n, a_0 = a_1 = 1$

Use formula 2a) and solve for c_1 and c_2.

We get, as roots, $r = 2$ and $r = 3$. So we have $a_n = c_1 2^n + c_2 3^n$. So, $c_1 + c_2 = 1$, and $2c_1 + 3c_2 = 1$.

Solving, we get $c_2 = 1$ and $c_1 = 2$. Therefore,

$$a_n = 2^{n+1} + 3^n$$

(b) (10 points) $a_{n+2} = 4a_{n+1} - 4a_n, a_0 = a_1 = 1$

Using formula 2b, we immediately get the answer. Details left as practice for you.

3. (a) (10 points) Prove $E(B_{n,p}) = np$.

The expacted value of $B_{1,p} = p$, as $0 * (1 - p) + 1 * p = p$.

By additivity of expectation, flipping n such coins gives us an answer of $B_{n,p} = np$.

(b) (10 points) Prove $\mathrm{Var}(B_{n,p}) = np(1 - p)$.

If X_i is the varaible with 1 if coin i is heads and 0 if coin i is tails, we know that the X_i are all independent. Also, we know that $B_{n,p} = \sum_{i=1}^{n} X_i$.

Therefore, $\mathrm{Var}\, B_{n,p} = nB_{1,p}$. And $B_{1,p} = E(B_{1,p}^2) - E_(B_{1,p})^2$, which equals $p - p^2 = p(1 - p)$. The result folllows.

4. (20 Points)

(a) (10 points) Suppose we roll two distinguishable fair n-**sided** dice. Let X be the **product** of the two numbers on top. What is $E(X)$, in terms of n?

Since dice are independent, we use the multiplicative rule for expectation. For any given die, the expected value is $(1 + 2 + \cdots + n)/n = (n + 1)/2$.

If we throw two such die, and multiply the totals together, the multiplicative rule tells us that the final answer is $\left(\frac{n+1}{2}\right)^2$.

(b) (10 points) Prove that $\Pr(40 \le B_{100,1/2} \le 60) \ge 3/4$.

By 3b, $B_{100,1/2} = 25$, so the standard deviation is just 5. The above formula is the probability that $B_{100,1/2}$ deviates from 50, it's expectation, by at most 10, which is twice the standard diviation. By Chebyshev's inequality, varying by *more* than twice the standard deviation can happen with probability at most 1/4, and the result follows.

5. Let X and Y be random variables over Ω. Prove or disprove: If $E(XY) = E(X)E(Y)$, then X and Y are independent.

The statement is false.

Let X take the values -1, 0, and 1, with equal probability.

Let $Y = 0$ if $X \neq 0$ and 1 otherwise.

Then $E(X) = 0$, so $E(X)E(Y) = 0$. Also, $XY = 0$ always, so $E(XY) = 0$.

However, $\Pr(Y \geq 1/2) = 2/3$, but $\Pr(Y \geq 1/2 | X \geq 0) = 1/2$, so X and Y are not independent.

Formula Sheet:

1. If $a_{n+2} + b_1 a_{n+1} + b_2 a_{n+2} = 0$:
   (a) If the roots for $x^2 + b_1 x + b_2$ are r_1, r_2, where r_1, r_2 are *distinct* real numbers, then $a_n = c_1 r_1^n + c_2 r_2^n$.
   (b) If there is one real root r, then $a_n = a_0 r^n + \frac{a_1 - a_0 r}{r} n r^n$
2. $\Pr(A|B) = \frac{\Pr(A \wedge B)}{\Pr(B)}$
3. If X is a random variable:
$$E(X) = \sum_{k \in \mathbb{R}} \Pr(X = k) = \sum_{\omega \in \Omega} X(\omega) \Pr(\omega).$$
4. $\mathrm{Var}(X) = E\left((X - E(X))^2\right) = E(X^2) - (E(X))^2$.
5. The *standard deviation* $\sigma(X)$ of X is $\sqrt{\mathrm{Var}(X)}$.
6. Let $X_1, \ldots, X_n$ be random variables over Ω:
   (a) $E(\sum_{i=1}^n X_i) = \sum_{i=1}^n E(X_i)$
   (b) If, for $i \neq j$, we know X_i and X_j are independent, then $\mathrm{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathrm{Var}(X_i)$
   (c) If X and Y are independent random variables, $E(XY) = E(X)E(Y)$.
7. Suppose we flip a coin where heads has probability p n times. Then $B_{n,p}$ is the random variable representing the number of heads obtained.
8. Law of large numbers: $\lim_{n \to \infty} \Pr(|B_{n,p} - np| \geq \epsilon np) = 0$.
9. Chebyshev inequality: $\Pr(|X - \mu| \geq t\sigma(X)) \leq \frac{1}{t^2}$.
10. Markov Inequality: If X is a random variable that takes only *nonnegative* values, then $\Pr(X \geq t) \leq \frac{E(X)}{t}$.