Random Variables

A function $Z : \Omega \to \mathbf{R}$ is called a random variable.

Two Dice

$Z(x_1, x_2) = x_1 + x_2$.
$p_k = \mathbf{P}(Z = k) = \mathbf{P}(\{\omega : \ Z(\omega) = k\})$.

k	2	3	4	5	6	7	8	9	10	11	12
p_k	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Coloured Balls

$\Omega = \{k$ indistinguishable balls, n colours $\}$.

Uniform distribution.

$Z =$ no. colours used.

$$p_m = \frac{\binom{n}{m}\binom{k-1}{m-1}}{\binom{n+k-1}{k}}.$$

If $k = 10, n = 5$ then

$p_1 = \frac{5}{1001}, p_2 = \frac{90}{1001}, p_3 = \frac{360}{1001}, p_4 = \frac{420}{1001},$

$p_5 = \frac{126}{1001}.$

Binomial Random Variable $B_{n,p}$.

n coin tosses. $p = \mathbf{P}(Heads)$ for each toss.
$\Omega = \{H, T\}^n$.

$$\mathbf{P}(\omega) = p^k(1-p)^{n-k}$$

where k is the number of H's in ω.
$B_{n,p}(\omega)$ = no. of occurrences of H in ω.

$$\mathbf{P}(B_{n,p} = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

If $n = 8$ and $p = 1/3$ then

$p_0 = \frac{2^8}{3^8}$, $p_1 = 8 \times \frac{2^7}{3^8}$, $p_2 = 28 \times \frac{2^6}{3^8}$,

$p_3 = 56 \times \frac{2^5}{3^8}$, $p_4 = 140 \times \frac{2^4}{3^8}$, $p_5 = 56 \times \frac{2^3}{3^8}$,

$p_6 = 28 \times \frac{2^2}{3^8}$, $p_7 = 8 \times \frac{2}{3^8}$, $p_8 = \frac{1}{3^8}$

Poisson Random Variable $Po(\lambda)$.

$\Omega = \{0, 1, 2, \ldots,\}$ and

$$\mathbf{P}(Po(\lambda) = k) = \frac{\lambda^k e^{-\lambda}}{k!} \qquad \text{for all } k \geq 0.$$

This is a limiting case of $B_{n,\lambda/n}$ where $n \to \infty$.

$Po(\lambda)$ is the number of occurrences of an event which is individually rare, but has constant expectation in a large population.

Fix k, then

$$\begin{aligned}\lim_{n\to\infty} \mathbf{P}(B_{n,\lambda/n} = k) &= \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{\lambda^k e^{-\lambda}}{k!}\end{aligned}$$

Explanation of $\binom{n}{k} \approx n^k/k!$ for fixed k.

$$\begin{aligned}\frac{n^k}{k!} &\geq \binom{n}{k} \\ &= \frac{n^k}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) \\ &\geq \frac{n^k}{k!}\left(1 - \frac{k(k-1)}{2n}\right)\end{aligned}$$

Expectation (Average)

Z is a random variable. Its *expected value* is given by

$$\begin{aligned}\mathbf{E}(Z) &= \sum_{\omega\in\Omega} Z(\omega)\mathbf{P}(\omega) \\ &= \sum_{k} k\mathbf{P}(Z=k).\end{aligned}$$

Ex: **Two Dice**

$Z = x_1 + x_2$.

$$\mathbf{E}(Z) = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + \cdots + 12 \times \frac{1}{36} = 7.$$

10 indistinguishable balls, 5 colours. Z is the number of colours actually used.

$$\mathbf{E}(Z) = \frac{5}{1001} + 2 \times \frac{90}{1001} + 3 \times \frac{360}{1001} + 4 \times \frac{420}{1001} + 5 \times \frac{126}{1001}.$$

In general: n colours, m balls.

$$\begin{aligned}
\mathbf{E}(Z) &= \sum_{k=1}^{n} \frac{k\binom{n}{k}\binom{m-1}{k-1}}{\binom{n+m-1}{m}} \\
&= n \sum_{k=1}^{n} \frac{\binom{n-1}{k-1}\binom{m-1}{k-1}}{\binom{n+m-1}{m}} \\
&= n \sum_{k-1=0}^{n-1} \frac{\binom{n-1}{k-1}\binom{m-1}{m-k}}{\binom{n+m-1}{m}} \\
&= \frac{n\binom{n+m-2}{m-1}}{\binom{n+m-1}{m}} \\
&= \frac{mn}{n+m-1}.
\end{aligned}$$

Geometric

$\Omega = \{1, 2, \dots, \}$
$\mathbf{P}(k) = (1-p)^{k-1}p$, $Z(k) = k$.

$$\begin{aligned}\mathbf{E}(Z) &= \sum_{k=1}^{\infty} k(1-p)^{k-1}p \\ &= \frac{p}{(1-(1-p))^2} \\ &= \frac{1}{p}\end{aligned}$$

$=$ expected number of trials until success.

$$\left[\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}.\right]$$

Binomial $B_{n,p}$.

$$
\begin{aligned}
\mathbf{E}(B_{n,p}) &= \sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k} \\
&= \sum_{k=1}^{n} n\binom{n-1}{k-1}p^k(1-p)^{n-k} \\
&= np\sum_{k=1}^{n} \binom{n-1}{k-1}p^{k-1}(1-p)^{n-k} \\
&= np(p+(1-p))^{n-1} \\
&= np.
\end{aligned}
$$

Poisson $Po(\lambda)$.

$$
\begin{aligned}
\mathbf{E}(Po(\lambda)) &= \sum_{k=0}^{\infty} k\frac{\lambda^k e^{-\lambda}}{k!} \\
&= \lambda\sum_{k=1}^{\infty} \frac{\lambda^{k-1}e^{-\lambda}}{(k-1)!} \\
&= \lambda.
\end{aligned}
$$

Suppose X, Y are random variables on the same probability space Ω.

Claim: $\mathbf{E}(X+Y) = \mathbf{E}(X) + \mathbf{E}(Y)$.
Proof:

$$\begin{aligned}
E(X+Y) &= \sum_{\alpha}\sum_{\beta}(\alpha+\beta)\mathbf{P}(X=\alpha, Y=\beta) \\
&= \sum_{\alpha}\sum_{\beta}\alpha\mathbf{P}(X=\alpha, Y=\beta) + \sum_{\alpha}\sum_{\beta}\beta\mathbf{P}(X=\alpha, Y=\beta) \\
&= \sum_{\alpha}\alpha\sum_{\beta}\mathbf{P}(X=\alpha, Y=\beta) + \sum_{\beta}\beta\sum_{\alpha}\mathbf{P}(X=\alpha, Y=\beta) \\
&= \sum_{\alpha}\alpha\mathbf{P}(X=\alpha) + \sum_{\beta}\beta\mathbf{P}(Y=\beta) \\
&= \mathbf{E}(X) + \mathbf{E}(Y).
\end{aligned}$$

In general if $X_1, X_2, \ldots, X_n$ are random variables on Ω then

$$\mathbf{E}(X_1+X_2+\cdots+X_n) = \mathbf{E}(X_1)+\mathbf{E}(X_2)+\cdots+\mathbf{E}(X_n)$$

Binomial

Write $B_{n,p} = X_1 + X_2 + \cdots + X_n$ where $X_i = 1$ if the ith coin comes up heads.

$$\mathbf{E}(B_{n,p}) = \mathbf{E}(X_1) + \mathbf{E}(X_2) + \cdots + \mathbf{E}(X_n) = np$$

since $\mathbf{E}(X_i) = p \times 1 + (1-p) \times 0$.

Same probability space. $Z(\omega)$ denotes the number of occurrences of the sequence H, T, H in ω.
$Z = X_1 + X_2 + \cdots + X_{n-2}$ where $X_i = 1$ if coin tosses $i, i+1, i+2$ come up H, T, H respectively. So

$$\mathbf{E}(Z) = \mathbf{E}(X_1) + \mathbf{E}(X_2) + \cdots + \mathbf{E}(X_{n-2}) = (n-2)p^2(1-p),$$

since $\mathbf{P}(x_i = 1) = p^2(1-p)$.

m **indistinguishable** balls, n colours. Z is the number of colours actually used.

$Z_i = 1 \;\leftrightarrow$ colour i is used.
$Z = Z_1 + \cdots + Z_n$= number of colours actually used.

$$\begin{aligned}
\mathbf{E}(Z) &= \mathbf{E}(Z_1) + \cdots + \mathbf{E}(Z_n) \\
&= n\mathbf{E}(Z_1) \\
&= n \Pr(Z_1 \neq 0) \\
&= n\left(1 - \frac{\binom{n+m-2}{m}}{\binom{n+m-1}{m}}\right). \\
&= n\left(1 - \frac{n-1}{n+m-1}\right) \\
&= \frac{mn}{n+m-1}.
\end{aligned}$$

m **distinguishable balls,** n **boxes**

$$\begin{aligned} Z &= \text{number of non-empty boxes.} \\ &= Z_1 + Z_2 + \cdots + Z_n \end{aligned}$$

where $Z_i = 1$ if box i is non-empty and $= 0$ otherwise. Hence,

$$\mathbf{E}(Z) = n\left(1 - \left(1 - \frac{1}{n}\right)^m\right),$$

since $\mathbf{E}(Z_i) = \mathbf{P}($ box i is non-empty$) =$ $\left(1 - \left(1 - \frac{1}{n}\right)^m\right)$.

Why is this different from the previous slide? The answer is that the indistinguishable balls space is obtained by partitioning the distinguishable balls space and then giving each set of the partition equal probability as opposed to a probability proportional to its size.
For example, if the balls are indistinguishable then the probability of exactly one non-empty box is $n \times \binom{m+n-1}{n-1}^{-1}$ whereas, if the balls are distinguishable, this probability becomes $n \times n^{-m}$.

Conditional Expectation

Suppose $A \subseteq \Omega$ and Z is a a random variable on Ω. Then

$$\mathbf{E}(Z \mid A) = \sum_{\omega \in A} Z(\omega)\mathbf{P}(\omega \mid A) = \sum_{k} k\mathbf{P}(Z = k \mid A).$$

Ex: **Two Dice**

$Z = x_1 + x_2$ and $A = \{x_1 \geq x_2 + 4\}$.
$A = \{(5,1),(6,1),(6,2)\}$ and so $\mathbf{P}(A) = 1/12$.

$$\mathbf{E}(Z \mid A) = 6 \times \frac{1/36}{1/12} + 7 \times \frac{1/36}{1/12} + 8 \times \frac{1/36}{1/12} = 7.$$

Let $B_1, B_2, \ldots, B_n$ be pairwise disjoint events which partition Ω. Let Z be a random variable on Ω. Then

$$\mathbf{E}(Z) = \sum_{i=1}^{n} \mathbf{E}(Z \mid B_i)\, \mathsf{Pr}(B_i).$$

Proof

$$\begin{aligned}
\sum_{i=1}^{n} \mathbf{E}(Z \mid B_i)\mathbf{P}(B_i) &= \sum_{i=1}^{n} \sum_{\omega \in B_i} Z(\omega) \frac{\mathbf{P}(\omega)}{\mathbf{P}(B_i)} \mathbf{P}(B_i) \\
&= \sum_{i=1}^{n} \sum_{\omega \in B_i} Z(\omega)\mathbf{P}(\omega) \\
&= \sum_{\omega \in \Omega} Z(\omega)\mathbf{P}(\omega) \\
&= \mathbf{E}(Z).
\end{aligned}$$

Hashing

Let $U = \{0, 1, \ldots, N-1\}$ and $H = \{0, 1, \ldots, n-1\}$ where n divides N and $N \gg n$. $f : U \to H$, $f(u) = u \mod n$.
(H is a hash table and U is the universe of objects from which a subset is to be stored in the table.)

Suppose $u_1, u_2, \ldots, u_m$, $m = \alpha n$, are a random subset of U. A copy of u_i is stored in "cell" $f(u_i)$ and u_i's that "hash" to the same cell are stored as a linked list.

Questions: u is chosen uniformly from U.
(i) What is the expected time T_1 to determine whether or not u is in the table?
(ii) If it is given that u is in the table, what is the expected time T_2 to find where it is placed?

Time = The number of comparisons between elements of U needed.

Let $M = N/n$, the number of u's that map to a cell. Let X_k denote the number of u_i for which $f(u_i) = k$. Then

$$\begin{aligned}
\mathbf{E}(T_1) &= \sum_{k=1}^{n} \mathbf{E}(T_1 \mid f(u) = k)\mathbf{P}(f(u) = k) \\
&= \frac{1}{n} \sum_{k=1}^{n} \mathbf{E}(T_1 \mid f(u) = k) \\
&= \frac{1}{n} \sum_{k=1}^{n} \mathbf{E}\left(\frac{1 + X_k}{2}\frac{X_k}{M} + X_k\left(1 - \frac{X_k}{M}\right)\right) \\
&\leq \frac{1}{n} \sum_{k=1}^{n} \mathbf{E}(X_k) \\
&= \frac{1}{n}\mathbf{E}\left(\sum_{k=1}^{n} X_k\right) \\
&= \alpha.
\end{aligned}$$

Let X denote $X_1, X_2, \ldots, X_n$ and let $\mathcal{X}$ denote the set of possible values for X. Then

$$
\begin{aligned}
\mathbf{E}(T_2) &= \sum_{X \in \mathcal{X}} \mathbf{E}(T_2 \mid X)\mathbf{P}(X) \\
&= \sum_{X \in \mathcal{X}} \sum_{k=1}^{n} \mathbf{E}(T_2 \mid f(u) = k, X) \\
&\qquad\qquad\qquad \times \mathbf{P}(f(u) = k)\mathbf{P}(X) \\
&= \sum_{X \in \mathcal{X}} \sum_{k=1}^{n} \mathbf{E}(T_2 \mid f(u) = k, X)\frac{X_k}{m}\mathbf{P}(X) \\
&= \sum_{X \in \mathcal{X}} \sum_{k=1}^{n} \left(\frac{1 + X_k}{2}\right) \frac{X_k}{m}\mathbf{P}(X) \\
&= \frac{1}{2m} \sum_{X \in \mathcal{X}} \sum_{k=1}^{n} X_k(1 + X_k)\mathbf{P}(X) \\
&= \frac{1}{2} + \frac{1}{2M}\mathbf{E}(X_1^2 + \cdots + X_n^2) \\
&= \frac{1}{2} + \frac{1}{2\alpha}\mathbf{E}(X_1^2) \\
&= \frac{1}{2} + \frac{1}{2\alpha} \sum_{t=1}^{m} t^2 \frac{\binom{M}{t}\binom{N-M}{m-t}}{\binom{N}{m}}.
\end{aligned}
$$

If α is small and t is small then we can write

$$\frac{\binom{M}{t}\binom{N-M}{m-t}}{\binom{N}{m}} \approx \frac{M^t}{t!}\frac{(N-M)^{m-t}}{(m-t)!}\frac{m!}{N^m}$$

$$\approx \left(1-\frac{1}{n}\right)^m \frac{m^t}{t!n^t} \approx \frac{\alpha^t e^{-\alpha}}{t!}.$$

Then we can further write

$$\mathbf{E}(T_2) \approx \frac{1}{2} + \frac{1}{2\alpha}\sum_{t=1}^{\infty} t^2 \frac{\alpha^t e^{-\alpha}}{t!} = 1 + \frac{\alpha}{2}$$

Random Walk: Suppose we do n steps of previously described random walk. Let Z_n denote the number of times the walk visits the origin. Then

$$Z_n = Y_0 + Y_1 + Y_2 + \cdots + Y_n$$

where $Y_i = 1$ if $X_i = 0$ – recall that X_i is the position of the particle after i moves.

But

$$\mathbf{E}(Y_i) = \begin{cases} 0 & i \text{ odd} \\ \binom{i}{i/2} 2^{-i} & i \text{ even} \end{cases}$$

So

$$\begin{aligned} \mathbf{E}(Z_n) &= \sum_{\substack{0 \le m \le n \\ m \ even}} \binom{m}{m/2} 2^{-m}. \\ &\approx \sum \sqrt{2/(\pi m)} \\ &\approx \frac{1}{2} \int_0^n \sqrt{2/(\pi x)} dx \\ &= \sqrt{2n/\pi} \end{aligned}$$

Consider the following program which computes the minimum of the n numbers $x_1, x_2, \ldots, x_n$.

begin
$min := \infty$;
for $i = 1$ **to** n **do**
begin
if $x_i < min$ **then** $min := x_i$
end
output min
end

If the x_i are all different and in random order, what is the expected number of times that that the statement $min := x_i$ is executed?

Ω = {permutations of $1, 2, \ldots, n$} – uniform distribution.

Let X be the number of executions of statement $min := x_i$. Let

$$X_i = \begin{cases} 1 & \text{statement executed at } i. \\ 0 & \text{otherwise} \end{cases}$$

Then $X_i = 1$ iff $x_i = \min\{x_1, x_2, \ldots, x_i\}$ and so

$$\mathbf{P}(X_i = 1) = \frac{(i-1)!}{i!} = \frac{1}{i}.$$

[The number of permutations of $\{x_1, x_2, \ldots, x_i\}$ in which x_i is the largest is $(i-1)!$.] So

$$\begin{aligned} \mathbf{E}(X) &= \mathbf{E}\left(\sum_{i=1}^{n} X_i\right) \\ &= \sum_{i=1}^{n} \mathbf{E}(X_i) \\ &= \sum_{i=1}^{n} \frac{1}{i} \qquad (= \; H_n) \\ &\approx \log_e n. \end{aligned}$$

Independent Random Variables

Random variables X, Y defined on the same probability space are called independent if for all α, β the events $\{X = \alpha\}$ and $\{Y = \beta\}$ are independent.

Example: if $\Omega = \{0, 1\}^n$ and the values of X, Y depend only on the values of the bits in disjoint sets Δ_X, Δ_Y then X, Y are independent.

E.g. if X = number of 1's in first m bits and Y = number of 1's in last $n - m$ bits.

The independence of X, Y follows directly from the disjointness of $\Delta_{\{X=\alpha\}}$ and $\Delta_{\{Y=\beta\}}$.

If X and Y are **independent** random variables then

$$\mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y).$$

$$\begin{aligned}
\mathbf{E}(XY) &= \sum_\alpha \sum_\beta \alpha\beta \mathbf{P}(X=\alpha, Y=\beta) \\
&= \sum_\alpha \sum_\beta \alpha\beta \mathbf{P}(X=\alpha)\mathbf{P}(Y=\beta) \\
&= \left[\sum_\alpha \alpha \mathbf{P}(X=\alpha)\right]\left[\sum_\beta \beta \mathbf{P}(Y=\beta)\right] \\
&= \mathbf{E}(X)\mathbf{E}(Y).
\end{aligned}$$

This is not true if X and Y are not independent. E.g. Two Dice: $X = x_1 + x_2$ and $Y = x_1$. $\mathbf{E}(X) = 7$, $\mathbf{E}(Y) = 7/2$ and $\mathbf{E}(XY) = \mathbf{E}(x_1^2) + \mathbf{E}(x_1x_2) = 91/6 + (7/2)^2$.

If $X = B_{n,p}$ = number of heads in n coin flips and $Y = n - B_{n,p}$ then X and Y are not independent. E.g. $\mathbf{P}(X = n) = p^n$ but
$\mathbf{P}(X = n \mid Y = n) = 0$.

Now suppose the number of coin flips is the random variable $N = Po(\lambda)$. Let X be number of heads and Y be the number of tails. Let $q = 1 - p$.

$$\begin{aligned}
\mathbf{P}(X = x, Y = y) &= \mathbf{P}(X = x, Y = y \mid N = x + y) \\
&\qquad \times \mathbf{P}(N = x + y) \\
&= \binom{x+y}{x} p^x q!^y \frac{\lambda^{x+y}}{(x+y)!} e^{-\lambda} \\
&= \frac{(\lambda p)^x (\lambda q)^y}{x!y!} e^{-\lambda}.
\end{aligned}$$

$$
\begin{aligned}
\mathbf{P}(X=x) &= \sum_{n\geq x}\mathbf{P}(X=x \mid N=n)\mathbf{P}(N=n)\\
&= \sum_{n\geq x}\binom{n}{x}p^x q^{n-x}\frac{\lambda^n}{n!}e^{-\lambda}\\
&= \frac{(\lambda p)^x}{x!}e^{-\lambda}\sum_{n-x\geq 0}\frac{(\lambda q)^{n-x}}{(n-x)!}\\
&= \frac{(\lambda p)^x}{x!}e^{-\lambda}e^{\lambda q}\\
&= \frac{(\lambda p)^x}{x!}e^{-\lambda p}.
\end{aligned}
$$

Similarly,

$$\mathbf{P}(Y=y)=\frac{(\lambda q)^y}{y!}e^{-\lambda q}$$

and so

$$\mathbf{P}(X=x,Y=y)=\mathbf{P}(X=x)\mathbf{P}(Y=y)$$

for all x, y and the two random variables are independent!