Department of Mathematical Sciences
Carnegie Mellon University

Fall 2001

21-121 Calculus 1 (IM/Econ)

b

Exam #4

NAME: ____________________

Please circle your recitation section.

A	A. Winger	TuTh 12:30	PH A18C
B	A. Winger	TuTh 1:30	DH 1217
C	D. Berol	TuTh 3:30	DH 1211

- This exam consists of 6 problems. It is your responsibility to make sure you have all the pages.
- No notes or books may be consulted during the exam.
- No calculators may be used during the exam.
- All solutions that you submit must be your own work. You may not look at or copy the work of others during this exam.
- Show your work. No credit will be given for unsupported incorrect answers.

Problem	Points	Score
1	14	
2	24	
3	18	
4	14	
5	14	
6	16	
Total	100	

1. (14 points) Evaluate each of the following:

(a) $\displaystyle\int_0^1 \left(3e^{-3x} + \sqrt{x}\right)\,dx.$

Answer:

$$\int_0^1 \left(3e^{-3x} + x^{\frac{1}{2}}\right)\,dx = \left(-e^{-3x} + \frac{2}{3}x^{\frac{3}{2}}\right)\Bigg|_0^1 = \left(-e^{-3} + \frac{2}{3}\right) - \left(-e^{0}\right)$$

$$= \boxed{\frac{-1}{e^3} + \frac{5}{3}}$$

(b) $\displaystyle\int_{-1}^2 \left(2x + \sqrt{7} - 3x^2\right)\,dx.$

Answer:

$$\int_{-1}^2 \left(2x + \sqrt{7} - 3x^2\right)\,dx = \left(x^2 + \sqrt{7}x - x^3\right)\Big|_{-1}^2 = \left(4 + 2\sqrt{7} - 8\right) - \left(1 - \sqrt{7} + 1\right)$$

$$= \boxed{3\sqrt{7} - 6}$$

2. (24 points) Evaluate each of the following:

(a) $\int xe^{(x^2+5)}\,dx$

Answer: Let $u = x^2 + 5$, so $du = 2x\,dx$.

$$\int xe^{(x^2+5)}\,dx = \frac{1}{2}\int e^{(x^2+5)}(2x)\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C$$
$$= \boxed{\frac{1}{2}e^{(x^2+5)} + C}$$

(b) $\int\limits_0^1 x\sqrt{4x^2+5}\,dx$

Answer: Let $u = 4x^2 + 5$, so $du = 8x\,dx$.

$$\int\limits_0^1 x\sqrt{4x^2+5}\,dx = \frac{1}{8}\int\limits_0^1 \left(4x^2+5\right)^{\frac{1}{2}}(8x)\,dx = \frac{1}{8}\int\limits_{x=0}^{x=1} u^{\frac{1}{2}}\,du$$
$$= \left(\frac{1}{12}u^{\frac{3}{2}}\right)\Bigg|_{x=0}^{x=1} = \left(\frac{1}{12}\left(4x^2+5\right)^{\frac{3}{2}}\right)\Bigg|_0^1 = \left(\frac{27}{12}\right) - \left(\frac{5\sqrt{5}}{12}\right)$$
$$= \boxed{\frac{27-5\sqrt{5}}{12}}$$

(c) $\int \frac{3e^x}{e^x+4}\,dx$

Answer: Let $u = e^x + 4$, so $du = e^x\,dx$.

$$\int \frac{3e^x}{e^x+4}\,dx = 3\int \frac{1}{u}\,du = 3\ln|u| + C = 3\ln|e^x+4| + C$$
$$= \boxed{3\ln\left(e^x+4\right) + C}$$

3. (18 points) Evaluate each of the following:

(a) $\displaystyle\int \frac{x^3}{1+x^2}\,dx$

Answer: Let $u = 1 + x^2$, so $du = 2x\,dx$ and $x^2 = u - 1$.

$$\begin{aligned}\int x^3\left(1+x^2\right)^{-1}\,dx &= \frac{1}{2}\int x^2\left(1+x^2\right)^{-1}(2x)\,dx = \frac{1}{2}\int (u-1)u^{-1}\,du\\ &= \frac{1}{2}\int\left(1-u^{-1}\right)\,du = \frac{1}{2}\left(u - \ln|u|\right) + C\\ &= \boxed{\frac{1}{2}\left(1+x^2\right) - \frac{1}{2}\ln\left(1+x^2\right) + C}\end{aligned}$$

(b) $\displaystyle\frac{d}{dx}\left[\int_{e^x}^{x}\frac{1}{1+t^6}\,dt\right]$

Answer: Using the Fundamental Theorem of Calculus and the chain rule, we have

$$\begin{aligned}\frac{d}{dx}\left[\int_{e^x}^{x}\frac{1}{1+t^6}\,dt\right] &= \frac{d}{dx}\left[\int_{0}^{x}\frac{1}{1+t^6}\,dt - \int_{0}^{e^x}\frac{1}{1+t^6}\,dt\right]\\ &= \frac{1}{1+(x)^6}\frac{d}{dx}[x] - \frac{1}{1+(e^x)^6}\frac{d}{dx}[e^x]\\ &= \boxed{\frac{1}{1+x^6} - \frac{e^x}{1+e^{6x}}}\end{aligned}$$

4. (14 points) The demand equation for a product is

$$p = 9 - q^2,$$

and the supply equation is

$$p = 1 + q^2.$$

Determine the consumers' surplus and producers' surplus at market equilibrium.

Answer:
First, we find the point of market equilibrium.

$$9 - q^2 = 1 + q^2 \Rightarrow q^2 = 4 \Rightarrow q = \pm 2.$$

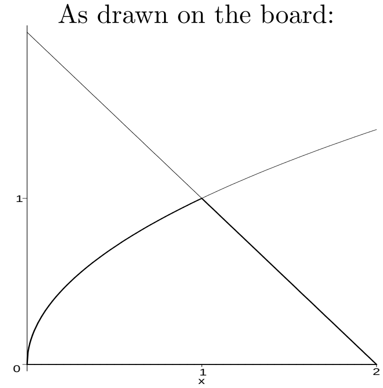
Since q must be non-negative, we find that the point at market equilibrium is $(2, 5)$.

$$\text{consumers' surplus} = \int_0^2 \left(9 - q^2 - 5\right)\, dq = \left(4q - \frac{1}{3}q^3\right)\Big|_0^2$$
$$= \left(8 - \frac{8}{3}\right) - 0 = \boxed{\frac{16}{3} \approx 5.33}$$

$$\text{producers' surplus} = \int_0^2 \left(5 - 1 - q^2\right)\, dq = \left(4q - \frac{1}{3}q^3\right)\Big|_0^2$$
$$= \left(8 - \frac{8}{3}\right) - 0 = \boxed{\frac{16}{3} \approx 5.33}$$

5. (14 points) Find the area of the region bounded by the curves $x = y^2$, $y = 2 - x$ and $y = 0$.

As drawn on the board:



Method 1 (vertical slices):

$$A = \int_0^1 \left[x^{\frac{1}{2}} - 0\right]\,dx + \int_1^2 [2 - x - 0]\,dx = \left(\frac{2}{3}x^{\frac{3}{2}}\right)\Big|_0^1 + \left(2x - \frac{1}{2}x^2\right)\Big|_1^2$$

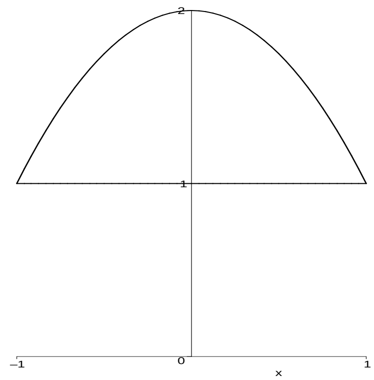
$$= \frac{2}{3} + 4 - 2 - 2 + \frac{1}{2} = \boxed{\frac{7}{6}}$$

Method 2 (horizontal slices):

$$A = \int_0^1 \left[(2 - y) - y^2\right]\,dy = \left(2y - \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_0^1$$

$$= 2 - \frac{1}{2} - \frac{1}{3} = \boxed{\frac{7}{6}}$$

6. (16 points) Find the volume of the solid generated when the region bounded by the curves $y = 2 - x^2$ and $y = 1$ is revolved about the x-axis.



Method 1 (vertical slices/washer method): The volume of each thin washer is

$$\left[\pi(\text{outer radius})^2 - \pi(\text{inner radius})^2\right]\,dx = \left[\pi\left(2-x^2\right)^2 - \pi(1)^2\right]\,dx$$
$$= \pi\left(x^4 - 4x^2 + 3\right)\,dx$$

Total volume is

$$V = \int_{-1}^{1} \pi\left(x^4 - 4x^2 + 3\right)\,dx = \pi\left(\frac{1}{5}x^5 - \frac{4}{3}x^3 + 3x\right)\Big|_{-1}^{1}$$
$$= \frac{\pi}{5} - \frac{4\pi}{3} + 3\pi + \frac{\pi}{5} - \frac{4\pi}{3} + 3 = \boxed{\frac{56\pi}{15}}$$

Method 2 (horizonatal slices/cylindrical shells): The volume of each cylindrical shell is

$$2\pi r h\,dy = 2\pi(y)\left[\sqrt{2-y} - \left(-\sqrt{2-y}\right)\right]\,dy = 4\pi y\sqrt{2-y}\,dy$$

Total volume is (let $u = 2 - y$, so $du = -dy$ and $y = 2 - u$)

$$V = \int_{1}^{2} 4\pi y\sqrt{2-y}\,dy = -4\pi\int_{y=1}^{y=2}(2-u)u^{\frac{1}{2}}\,du = \left(\frac{-16\pi}{3}u^{\frac{3}{2}} + \frac{8\pi}{5}u^{\frac{5}{2}}\right)\Big|_{y=1}^{y=2}$$
$$= \left(\frac{-16\pi}{3}(2-y)^{\frac{3}{2}} + \frac{8\pi}{5}(2-y)^{\frac{5}{2}}\right)\Big|_{1}^{2} = \frac{16\pi}{3} - \frac{8\pi}{5} = \boxed{\frac{56\pi}{15}}$$