Department of Mathematical Sciences
Carnegie Mellon University

Fall 2001

21-121 Calculus 1 (IM/Econ)

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Exam #4

NAME: ______________________________

Please circle your recitation section.

A	A. Winger	TuTh 12:30	PH A18C
B	A. Winger	TuTh 1:30	DH 1217
C	D. Berol	TuTh 3:30	DH 1211

- This exam consists of 6 problems. It is your responsibility to make sure you have all the pages.
- No notes or books may be consulted during the exam.
- No calculators may be used during the exam.
- All solutions that you submit must be your own work. You may not look at or copy the work of others during this exam.
- Show your work. No credit will be given for unsupported incorrect answers.

Problem	Points	Score
1	14	
2	24	
3	18	
4	14	
5	14	
6	16	
Total	100	

1. (14 points) Evaluate each of the following:

(a) $\displaystyle\int_{-1}^{2} \left(3x^2 - \sqrt{5} + x\right)\, dx.$

Answer:

$$\int_{-1}^{2} \left(3x^2 - \sqrt{5} + x\right)\, dx = \left(x^3 - \sqrt{5}x + \frac{1}{2}x^2\right)\Bigg|_{-1}^{2} = \left(8 - 2\sqrt{5} + 2\right) - \left(-1 + \sqrt{5} + \frac{1}{2}\right)$$

$$= \boxed{\frac{23}{2} - 3\sqrt{5}}$$

(b) $\displaystyle\int_{0}^{1} \left(\sqrt{x} - e^{-3x}\right)\, dx.$

Answer:

$$\int_{0}^{1} \left(x^{\frac{1}{2}} - e^{-3x}\right)\, dx = \left(\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{3}e^{-3x}\right)\Bigg|_{0}^{1} = \left(\frac{2}{3} + \frac{1}{3}e^{-3}\right) - \left(0 + \frac{1}{3}e^{0}\right)$$

$$= \boxed{1 + \frac{1}{e^3}}$$

2. (24 points) Evaluate each of the following:

(a) $\displaystyle\int_0^1 x^2\sqrt{2x^3+2}\,dx$

Answer: Let $u = 2x^3+2$, so $du = 6x^2\,dx$.

$$\int_0^1 x^2\left(2x^3+2\right)^{\frac{1}{2}}\,dx = \frac{1}{6}\int_0^1 \left(2x^3+2\right)^{\frac{1}{2}}\left(6x^2\right)\,dx = \frac{1}{6}\int_{x=0}^{x=1} u^{\frac{1}{2}}\,du$$

$$= \left(\frac{1}{9}u^{\frac{3}{2}}\right)\Big|_{x=0}^{x=1} = \left(\frac{1}{9}\left(2x^3+2\right)^{\frac{3}{2}}\right)\Big|_0^1 = \left(\frac{8}{9}\right) - \left(\frac{\sqrt{8}}{9}\right)$$

$$= \boxed{\frac{8-\sqrt{8}}{9}}$$

(b) $\displaystyle\int x^2 e^{(x^3-2)}\,dx$

Answer: Let $u = x^3-2$, so $du = 3x^2\,dx$.

$$\int x^2 e^{(x^3-2)}\,dx = \frac{1}{3}\int e^{(x^3-2)}\left(3x^2\right)\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C$$

$$= \boxed{\frac{1}{3}e^{(x^3-2)} + C}$$

(c) $\displaystyle\int \frac{3e^x}{\left(e^x+2\right)^2}\,dx$

Answer: Let $u = e^x+2$, so $du = e^x\,dx$.

$$\int \frac{3e^x}{\left(e^x+2\right)^2}\,dx = 3\int \left(e^x+2\right)^{-2}\left(e^x\right)\,dx = 3\int u^{-2}\,du = -3u^{-1} + C$$

$$= \boxed{\frac{-3}{e^x+2} + C}$$

3. (18 points) Evaluate each of the following:

(a) $\int x^3\sqrt{1+x^2}\,dx$

Answer: Let $u = 1+x^2$, so $du = 2x\,dx$ and $x^2 = u-1$.

$$\begin{aligned}\int x^3\left(1+x^2\right)^{\frac{1}{2}}\,dx &= \frac{1}{2}\int x^2\left(1+x^2\right)^{\frac{1}{2}}(2x)\,dx = \frac{1}{2}\int (u-1)u^{\frac{1}{2}}\,du\\ &= \frac{1}{2}\int\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du = \frac{1}{2}\left(\frac{2}{5}u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}\right)+C\\ &= \boxed{\frac{1}{5}\left(1+x^2\right)^{\frac{5}{2}}-\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+C}\end{aligned}$$

(b) $\dfrac{d}{dx}\left[\displaystyle\int_x^{x^3}\frac{1}{1+t^4}\,dt\right]$

Answer: Using the Fundamental Theorem of Calculus and the chain rule, we have

$$\begin{aligned}\frac{d}{dx}\left[\int_x^{x^3}\frac{1}{1+t^4}\,dt\right] &= \frac{d}{dx}\left[\int_0^{x^3}\frac{1}{1+t^4}\,dt-\int_0^{x}\frac{1}{1+t^4}\,dt\right]\\ &= \frac{1}{1+\left(x^3\right)^4}\frac{d}{dx}\left[x^3\right]-\frac{1}{1+(x)^4}\frac{d}{dx}[x]\\ &= \boxed{\frac{3x^2}{1+x^{12}}-\frac{1}{1+x^4}}\end{aligned}$$

4. (14 points) The demand equation for a product is

$$p = 9 - q^2,$$

and the supply equation is

$$p = 1 + q^2.$$

Determine the consumers' surplus and producers' surplus at market equilibrium.

Answer:
First, we find the point of market equilibrium.

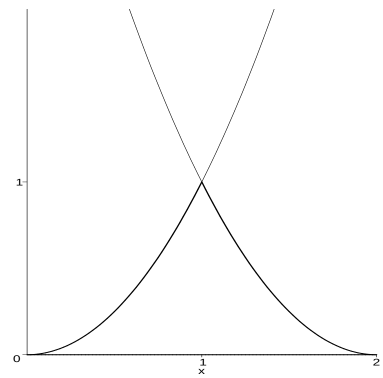
$$9 - q^2 = 1 + q^2 \Rightarrow q^2 = 4 \Rightarrow q = \pm 2.$$

Since q must be non-negative, we find that the point at market equilibrium is $(2, 5)$.

$$\begin{aligned} \text{consumers' surplus} &= \int_0^2 \left(9 - q^2 - 5\right)\, dq = \left(4q - \frac{1}{3}q^3\right)\Bigg|_0^2 \\ &= \left(8 - \frac{8}{3}\right) - 0 = \boxed{\frac{16}{3} \approx 5.33} \end{aligned}$$

$$\begin{aligned} \text{producers' surplus} &= \int_0^2 \left(5 - 1 - q^2\right)\, dq = \left(4q - \frac{1}{3}q^3\right)\Bigg|_0^2 \\ &= \left(8 - \frac{8}{3}\right) - 0 = \boxed{\frac{16}{3} \approx 5.33} \end{aligned}$$

5. (14 points) Find the area of the region bounded by the curves $y = x^2$, $y = (x-2)^2$ and $y = 0$.



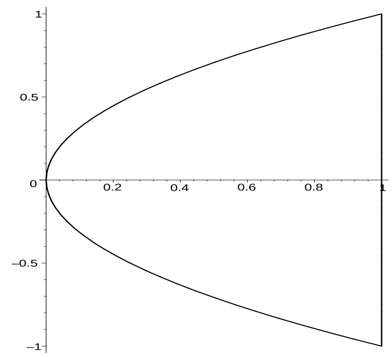
Method 1 (vertical slices):

$$A = \int_0^1 \left[x^2 - 0\right]\, dx + \int_1^2 \left[(x-2)^2 - 0\right]\, dx = \left(\frac{1}{3}x^3\right)\Big|_0^1 + \left(\frac{1}{3}x^3 - 2x^2 + 4x\right)\Big|_1^2$$
$$= \frac{1}{3} + \frac{8}{3} - 8 + 8 - \frac{1}{3} + 2 - 4 = \boxed{\frac{2}{3}}$$

Method 2 (horizontal slices):

$$A = \int_0^1 \left[\left(2 - y^{\frac{1}{2}}\right) - y^{\frac{1}{2}}\right]\, dy = \int_0^1 \left(2 - 2y^{\frac{1}{2}}\right)\, dy = \left(2y - \frac{4}{3}y^{\frac{3}{2}}\right)\Big|_0^1$$
$$= 2 - \frac{4}{3} = \boxed{\frac{2}{3}}$$

6. (16 points) Find the volume of the solid generated when the region bounded by the curves $x = y^2$ and $x = 1$ is revolved about the y-axis.



Method 1 (vertical slices/cylindrical shells): The volume of each cylindrical shell is

$$2\pi rh\,dx = 2\pi(x)\left[\sqrt{x} - \left(-\sqrt{x}\right)\right]\,dx = 4\pi x^{\frac{3}{2}}\,dx$$

Total volume is

$$V = \int_0^1 4\pi x^{\frac{3}{2}}\,dx = \left(\frac{8\pi}{5}x^{\frac{5}{2}}\right)\Big|_0^1$$
$$= \boxed{\frac{8\pi}{5}}$$

Method 2 (horizontal slices/washer method): The volume of each thin washer is

$$\left[\pi(\text{outer radius})^2 - \pi(\text{inner radius})^2\right]\,dy = \left[\pi(1)^2 - \pi\left(y^2\right)^2\right]\,dy = \pi\left(1 - y^4\right)\,dy$$

Total volume is

$$V = \int_{-1}^1 \pi\left(1 - y^4\right)\,dy = \left(\pi y - \frac{\pi}{5}y^5\right)\Big|_{-1}^1 = \left(\pi - \frac{\pi}{5}\right) - \left(-\pi + \frac{\pi}{5}\right)$$
$$= \boxed{\frac{8\pi}{5}}$$