Department of Mathematical Sciences
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21-121 Calculus 1 (IM/Econ)

Handout #3:
An Application of Integration to Economics

Suppose that the equation $p = f(q)$ provides the price p at which consumers will purchase (demand) q units. Further suppose that the equation $p = g(q)$ provides the price p at which the manufacturers will sell (supply) q units. The graph of $p = f(q)$ is called a demand curve, and the graph of $p = g(q)$ is called a supply curve. The point (q_0, p_0) where these two curves intersect (i.e. $p_0 = f(q_0) = g(q_0)$) is called the *point of equilibrium*. When the price for a product is p_0 per unit, the number of units demanded by the consumer is exactly equal to the number of units supplied by the manufacturer. Basically, the price p_0 is the price at which demand and supply are in perfect balance; in such a situation, we say that the market is at equilibrium.

Assume that the market is at equilibrium when the price per unit is p_0. When looking at a demand curve, one typically sees that there are consumers who would be willing to pay more than p_0 per unit. These consumers are receiving a benefit from the lower price p_0; they are spending less money per unit than they are actually willing to spend. The total gain to the consumers willing to pay more than p_0 can be represented by the quantity

$$\textbf{consumers' surplus} = \int_0^{q_0} [f(q) - p_0]\ dq,$$

where q_0 is such that $p_0 = f(q_0) = g(q_0)$. Geometrically, the consumers' surplus is the area below the demand curve and above the horizontal line $p = p_0$.

When looking at a supply curve, one notices that there are manufacturers that are willing to supply the product at a price less than p_0. These producers are benefitting from the higher equilibrium price p_0. The total gain for these producers can be represented by the quantity

$$\textbf{producers' surplus} = \int_0^{q_0} [p_0 - g(q)]\ dq.$$

Geometrically, the producers' surplus is the area above the supply curve and below the horizontal line $p = p_0$.

Let's look at an example. Suppose that the demand function for a product is given by

$$p = f(q) = 100 - 0.05q,$$

where p is the price per unit (in dollars) for q units. Suppose also that the supply function is

$$p = g(q) = 10 + 0.1q.$$

We want to find the consumers' surplus and the producers' surplus at market equilibrium.

The first thing to do is determine the point of equilibrium (q_0, p_0). This can be found by solving simultaneously the system of equations $p = 100 - 0.05q$ and $p = 10 + 0.1q$:

$$\begin{aligned} 100 - 0.05q &= 10 + 0.1q \\ 0.15q &= 90 \\ q &= 600. \end{aligned}$$

So $q_0 = 600$ and

$$p_0 = f(q_0) = g(q_0) = 70.$$

The point of equilibrium is $(70, 600)$.

Now, we use the formulas for the consumers' and producers' surplus. The consumers' surplus is

$$\begin{aligned} \int_0^{q_0} [f(q) - p_0]\, dq &= \int_0^{600} [(100 - 0.05q) - (70)]\, dq \\ &= \left(30q - \frac{0.05}{2}q^2\right)\Bigg|_0^{600} = 9000. \end{aligned}$$

The producers' surplus is

$$\begin{aligned} \int_0^{q_0} [p_0 - g(q)]\, dq &= \int_0^{600} [(70) - (10 + 0.1q)]\, dq \\ &= \left(60q - \frac{0.01}{2}q^2\right)\Bigg|_0^{600} = 18,000. \end{aligned}$$

So, the consumers' surplus is \$9000 and the consumers' surplus is \$18,000.