Concepts of Math: Recitations 28-29

December 8, 2015

Sets Review

1. Let A, B, C be arbitrary sets. If the equality below is true, prove it by double containment. If the equality below is false, show a counterexample.

$$A \times B - A \times C = A \times (B - C).$$

2. Let A, B, C, D be arbitrary sets. If the equality below is true, prove it by double containment. If the equality below is false, show a counterexample.

$$A \times B - C \times D = (A - C) \times (B - D).$$

3. The following problem was in Midterm 1.

   Let A and B be two arbitrary sets. Let $\mathcal{P}(A)$, $\mathcal{P}(B)$, and $\mathcal{P}(A - B)$ denote the power sets of A, B and $A - B$ respectively. Prove that

$$\mathcal{P}(A - B) - \mathcal{P}(\emptyset) \subseteq \mathcal{P}(A) - \mathcal{P}(B).$$

   (a) While proving the statement above, make it clear that subtracting $\mathcal{P}(\emptyset) = \{\emptyset\}$ on the left hand side is very important. The equality $\mathcal{P}(A - B) \subseteq \mathcal{P}(A) - \mathcal{P}(B)$ is false because $\emptyset \in \mathcal{P}(A - B)$, but $\emptyset \notin P(A) - \mathcal{P}(B)$.

   (b) Show that the inclusion below does not always hold.

$$\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B) - \mathcal{P}(\emptyset).$$

Problem 2 on Homework 10 hint (Irina's lecture)

If $x \in \mathbb{N}$, then

$$\frac{x(x+1)(x+2)\dots(x+k-1)}{k!} = \binom{x+k-1}{k}.$$

Induction Review

1. The following problem was in Midterm 1. Please solve it. Show why weak induction does not work here.

   Let $x \in \mathbb{R}$, $x \neq 0$, such that
   $$x + \frac{1}{x} \in \mathbb{Z}.$$
   Prove that for any integer $n \geq 0$ we have
   $$x^n + \frac{1}{x^n} \in \mathbb{Z}.$$

Logic Review

1. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. Without using words of negation, write the meaning of "f is not an increasing function".

2. In simpler language, describe the meaning of the following two statements and their negations. Which one implies the other, and why?

   (a) There is a number M such that, for every $x \in S$, $|x| \leq M$.

   (b) For every $x \in S$, there is a number M such that $|x| \leq M$.

3. Write the (full) DNF of $(p \leftrightarrow q) \wedge r$.

Bijections Review

1. Show an explicit bijection
   $$f : [0, 1] \cup [2, 3] \to [0, 3].$$
   Justify that your function is a bijection.

Combinatorics Review

1. Remind the students the following formulas. Give examples of non-trivial objects counted by each of these formulas.

   (a) The number of ways to place k differently colored marbles into n numbered boxes, at most one marble per box, is
   $$\frac{n!}{(n-k)!}.$$

(b) The number of ways to place k identical marbles into n numbered boxes, at most one marble per box, is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(c) The number of ways to place k differently colored marbles into n numbered boxes, any number of marbles per box, is

$$n^k.$$

(d) The number of ways to place k identical marbles into n numbered boxes, any number of marbles per box, is

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}.$$

(e) The number of ways to place k differently colored marbles into n numbered boxes, such that k_1 marbles are placed in the first box, k_2 marbles are placed in the second box, ..., k_n marbles are placed in the n^{th} box, and $k_1 + k_2 + \ldots + k_n = k$, is

$$\frac{k!}{k_1!k_2!\ldots k_n!}.$$

2. Find the coefficient of x^5 in the binomial expansion of $(x - 2x^{-2})^{20}$.

3. Find a simple expression for

$$3^n - \binom{n}{1}3^{n-1} + \ldots + (-1)^k\binom{n}{k}3^{n-k} + \ldots + (-1)^n.$$

Prove the resulting equality by counting a set in two ways (note that you will need to use the Principle of Inclusion-Exclusion).

4. Martina has three weeks to prepare for a tennis tournament. She decides to play at least one set every day but not more than 36 sets in all. Show that there is a period of consecutive days during which she will play exactly 21 sets.