Concepts of Math: Recitation 19

November 2, 2015

The Pigeonhole Principle

If you have not done it before, please solve the following problem.

1. A six by six checkerboard with 36 squares can be covered by exactly 18 dominoes. Prove that each such tiling can be cut by a horizontal or vertical line across the board without cutting any dominoes.

The Principle of Inclusion-Exclusion

Note that we have not done derangements yet. The first problem includes a solution with all the vocabulary and notations students should follow.

1. Given n sister-brother pairs, how many ways are there to form pairs consisting of one man and one woman who are not siblings?

   Solution. We will label the n sister-brother pairs with labels $1, 2, \ldots, n$. Let U be the set of ways to pair the men and women. Then $|U| = n!$. Let A_i be the set of ways to pair the men and women so that the ith sister and brother are paired. Then $U - A_1 \cup A_2 \cup \ldots \cup A_n$ is the set of ways to form pairs consisting of one man and one woman who are not siblings.

   Next we use the Inclusion-Exclusion Principle.

   $$|U - A_1 \cup A_2 \cup \ldots \cup A_n| =$$

   $$= |U| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \ldots + (-1)^n |A_1 \cup A_2 \cup \ldots A_n| =$$

   $$= n! - n(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \ldots + (-1)^n \binom{n}{n} 0! =$$

   $$= \sum_{k=0}^{n} (-1)^k \binom{n}{k}(n-k)!.$$

2. Three couples, the Smiths, Jonses, and Murphys, are going to form a line.

    (a) In how many such lines will Mr. and Mrs. Jones be next to each other?

    (b) In how many such lines will Mr. and Mrs. Jones be next to each other and Mr. and Mrs. Murphy be next to each other?

    (c) In how many such lines will at least one couple be next to each other? Use Inclusion-Exclusion here.

3. Let $n, m \in \mathbb{N}$ and $n \geq m$. Count the number of surjective functions from $[n]$ to $[m]$.

4. If there is time left, which is very unlikely, please answer a few homework questions.