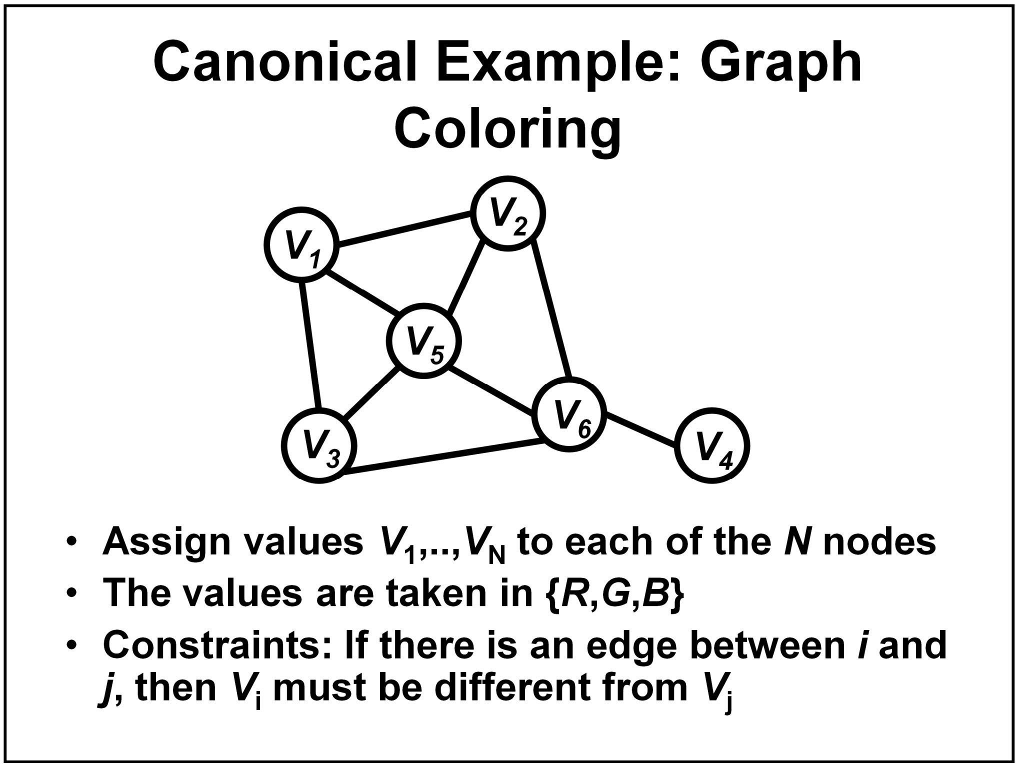
Constraint Satisfaction Problems

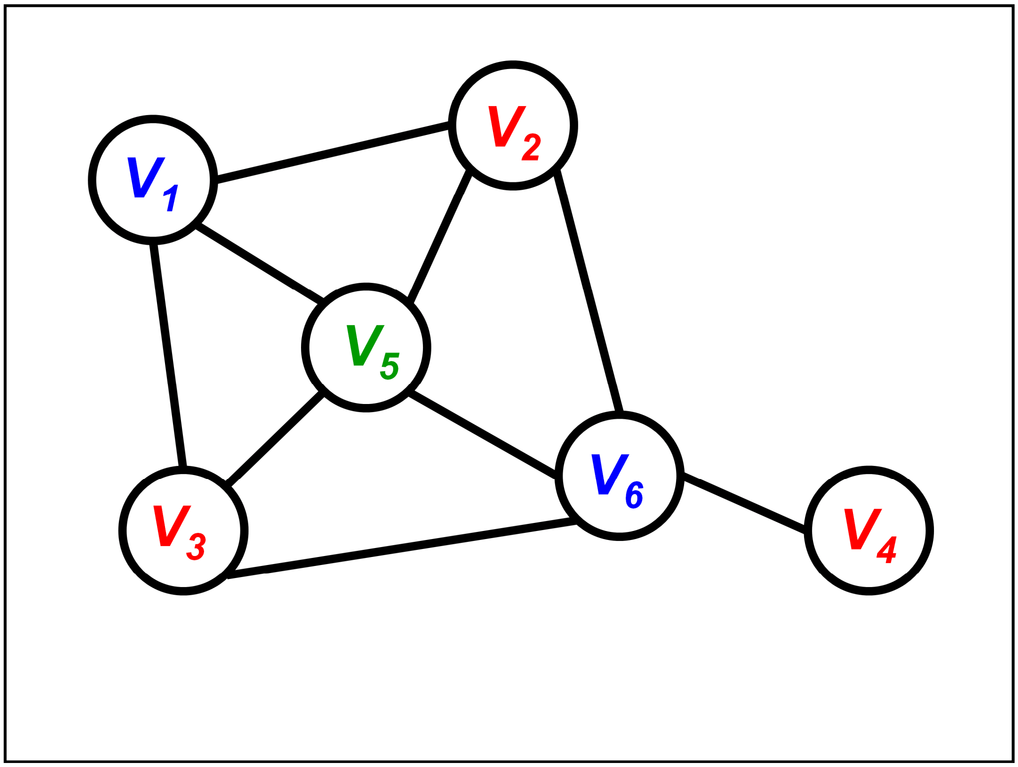


Canonical Example: Graph Coloring

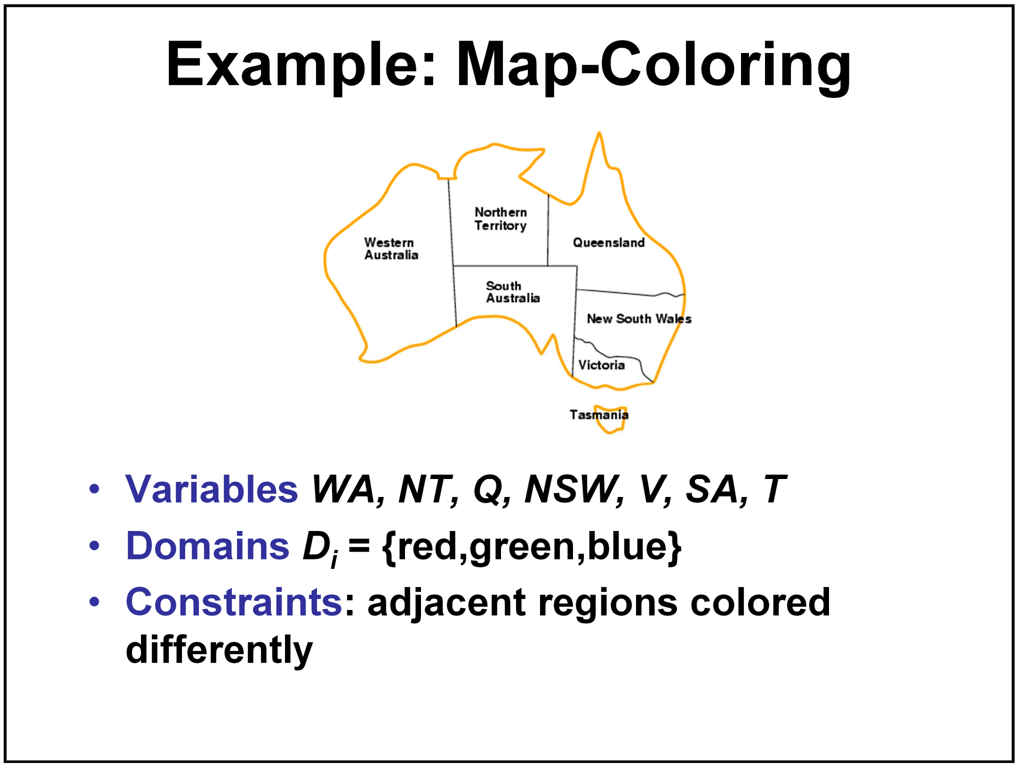
- **Assign values $V_1,..,V_N$ to each of the N nodes**
- **The values are taken in {R,G,B}**
- **Constraints: If there is an edge between i and j, then V_i must be different from V_j**

Constraint satisfaction basically boils down to graph coloring.

Graph coloring is NP complete, so any solution is *worst case* exponential time.



This is a valid 4-coloring…we're just not using the 4th color!



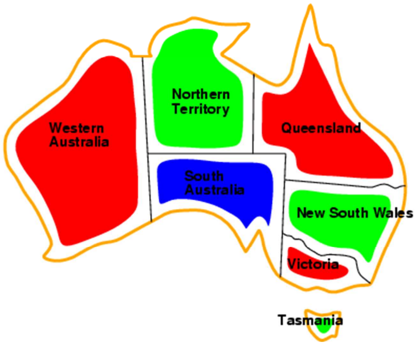
Example: Map-Coloring

- **Variables** ***WA, NT, Q, NSW, V, SA, T***
- **Domains** D_i **= {red,green,blue}**
- **Constraints: adjacent regions colored differently**

Note map coloring can be defined as graph coloring. Each state in map is represented as a node in the graph, with edges between nodes if the states are adjacent

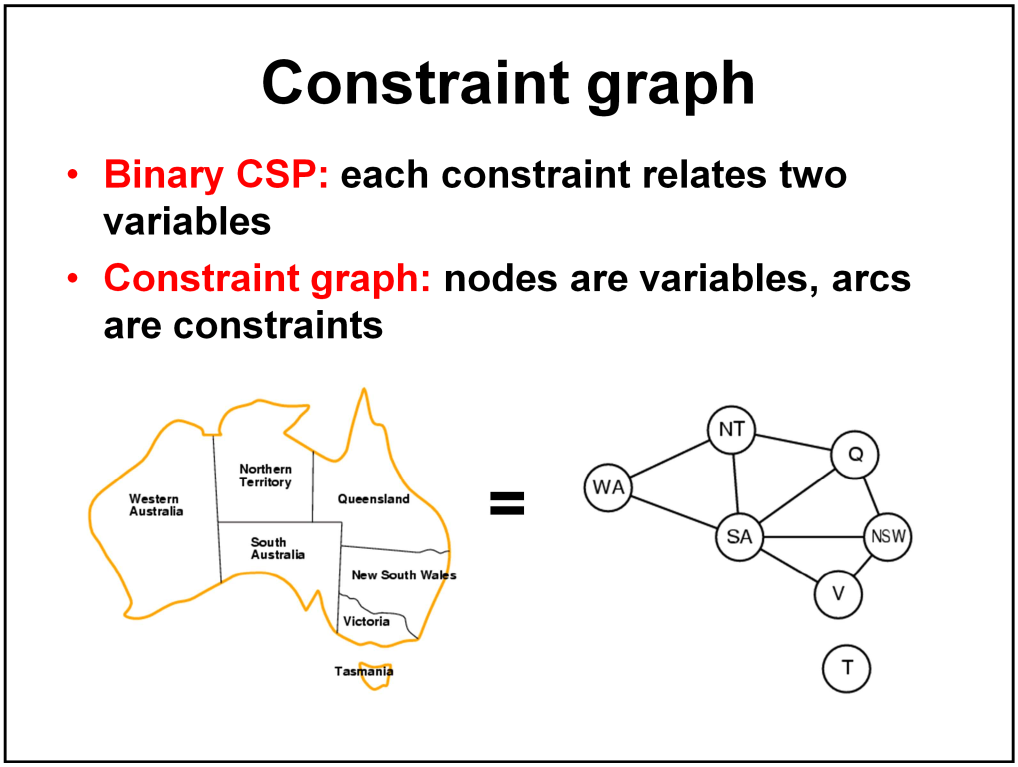
All map coloring can be represented as graph coloring, but not all graph coloring can be represented as map coloring. This is because maps are planar graphs.

Example: Map-Coloring



- **Solutions** are **complete** and **consistent** assignments, e.g.,
  - WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green

Complete: every state is colored. Consistent: it’s a valid soln.



Constraint graph

- **Binary CSP:** each constraint relates two variables
- **Constraint graph:** nodes are variables, arcs are constraints

Binary CSP can usually be expressed as a graph, because each edge has 2 endpoints, so each edge can be one constraint.

CSP Definition

- **CSP = {V, D, C}**
- ***Variables*: $V = \{V_1,..,V_N\}$**
- ***Domain*: The set of *d* values that each variable can take**
- ***Constraints*: $C = \{C_1,..,C_K\}$**
- **Each constraint consists of a tuple of variables and a list of values that the tuple is allowed to take for this problem**
  - **Example: $[(V_2,V_3),\{(R,B),(R,G),(B,R),(B,G),(G,R),(G,B)\}]$**
- **Constraints are usually defined implicitly**
  - **Example: $V_i \neq V_j$ for every edge (i,j)**

Example ([(V2,V3),{(R,B),…}]): the possible values the tuple (V2,V3) can take

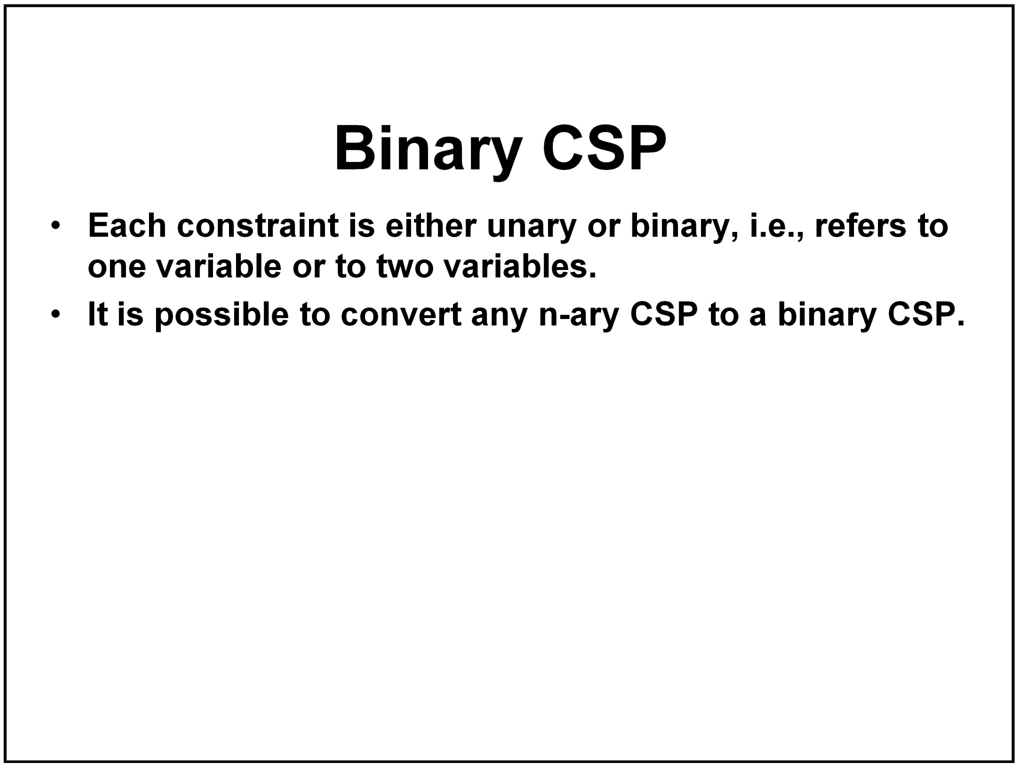
The second example is an easier way to write the first.

Varieties of constraints

- **Unary constraints involve a single variable,**
  - **e.g., SA ≠ green**
- **Binary constraints involve pairs of variables,**
  - **e.g., SA ≠ WA**
- **Higher-order constraints involve 3 or more variables**

Higher order example: allDif(X,Y,Z)

allDif(X,Y,Z) means X,Y, and Z all must be different.



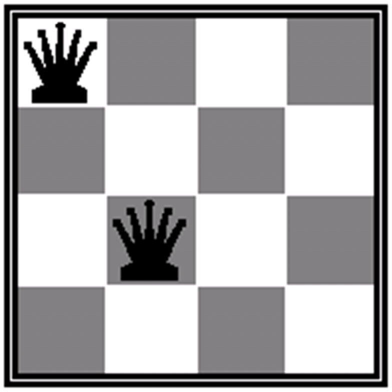
Binary CSP

- **Each constraint is either unary or binary, i.e., refers to one variable or to two variables.**
- **It is possible to convert any n-ary CSP to a binary CSP.**

Note you can convert any CSP to binary. Ex: diff(X,Y,Z) → diff(X,Y), diff(Y,Z), diff(Z,X)

When you convert to binary, # of constraints can blow up

N-Queens



$Q_1 = 1 \quad Q_2 = 3$

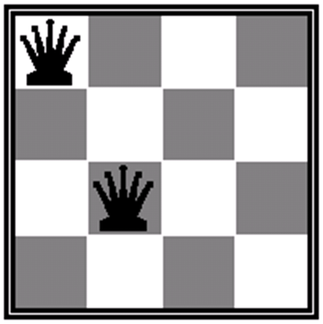
Given one queen per column, find row for each queen, such that there no queen attacks another queen.

i: column

Q_i: row

Example: N-Queens

- Variables: Q_i
- Domains: $D_i = \{1, 2, 3, 4\}$
- Constraints
  - $Q_i \neq Q_j$ (cannot be in the same row)
  - $|Q_i - Q_j| \neq |i - j|$ (or same diagonal)



$Q_1 = 1$ $Q_2 = 3$

- Valid values for (Q_1, Q_2) are (1,3) (1,4) (2,4) (3,1) (4,1) (4,2)

Cryptarithmetic

$$\begin{array}{r} \mathrm{S\,E\,N\,D} \\ \underline{+\,\mathrm{M\,O\,R\,E}} \\ \mathrm{M\,O\,N\,E\,Y} \end{array}$$

Example: Cryptarithmetic

- Variables
  D, E, M, N, O, R, S, Y
- Domains
  {0, 1, 2, 3 ,4, 5, 6, 7, 8, 9 }
- Constraints
  M ≠ 0, S ≠ 0 (unary constraints)
  Y = D + E OR Y = D + E – 10.
  D ≠ E, D ≠ M, D ≠ N, etc.

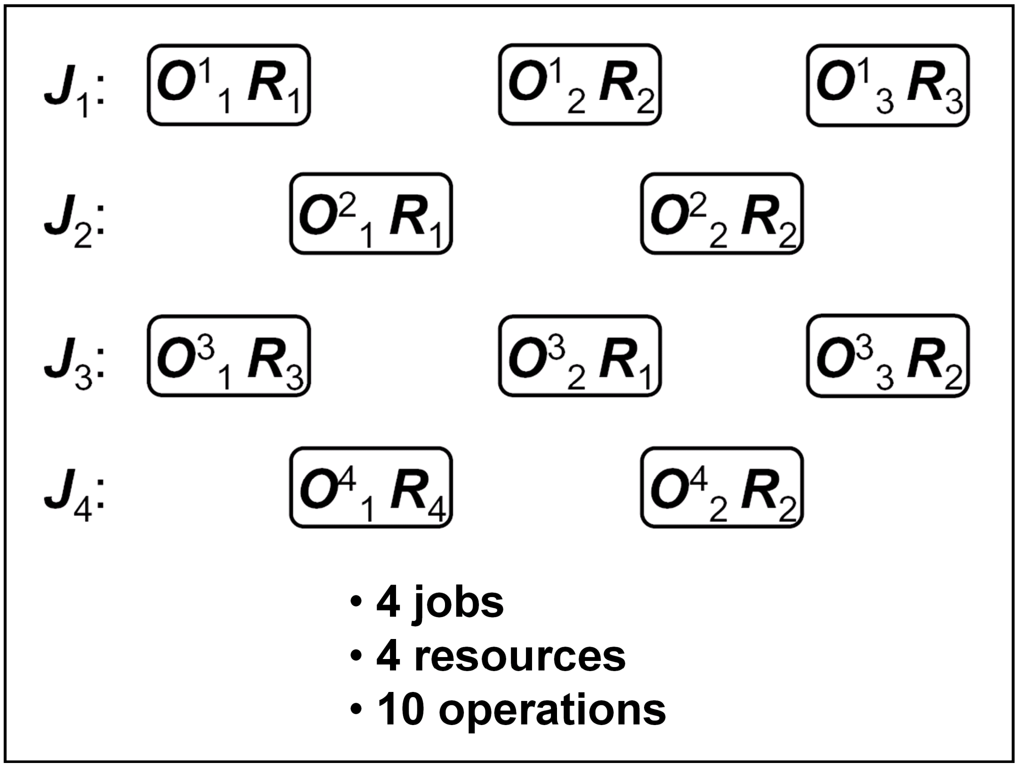
$$\begin{array}{r} \text{S E N D} \\ \underline{+\,\text{M O R E}} \\ \text{M O N E Y} \end{array}$$

Varieties of CSPs

- **Discrete variables**
  - **finite domains:**
    - ***n* variables, domain size *d* → *O(*d^n*)* complete assignments**
    - **e.g., Boolean CSPs, incl.~Boolean satisfiability (NP-complete)**
  - **infinite domains:**
    - **integers, strings, etc.**
    - **e.g., job scheduling, variables are start/end days for each job**
- **Continuous variables**
  - **e.g., start/end times for Hubble Space Telescope observations**
  - **linear constraints solvable in polynomial time by linear programming**

Example: Scheduling

- A set of N jobs, $J_1, \ldots, J_{n.}$
- Each job j is composed of a sequence of operations $O^j_1, \ldots, O^j_{Lj}$
- Each operation may use resource R, and has a specific duration in time.
- A resource must be used by a single operation at a time.
- All jobs must be completed by a due time.
- Problem: assign a start time to each job.



J_1: $O^1_1\ R_1$ $O^1_2\ R_2$ $O^1_3\ R_3$

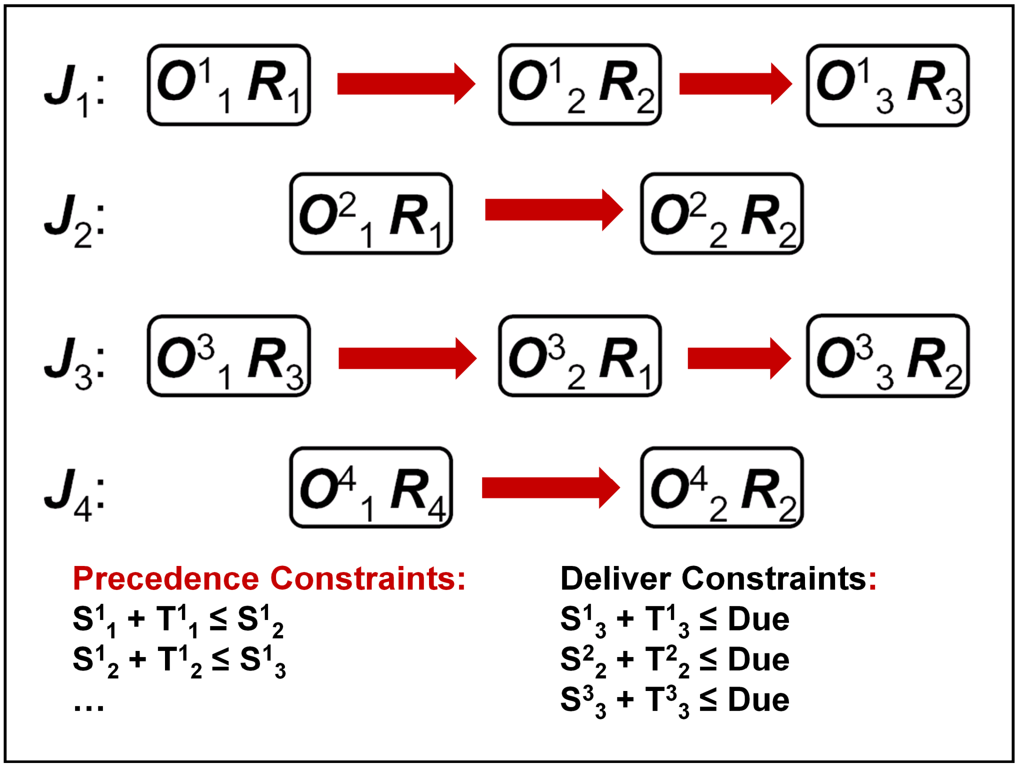
J_2: $O^2_1\ R_1$ $O^2_2\ R_2$

J_3: $O^3_1\ R_3$ $O^3_2\ R_1$ $O^3_3\ R_2$

J_4: $O^4_1\ R_4$ $O^4_2\ R_2$

- **4 jobs**
- **4 resources**
- **10 operations**

In addition to what we said before, need to wait for O^i_1 to finish before O^i_2 can start, etc.



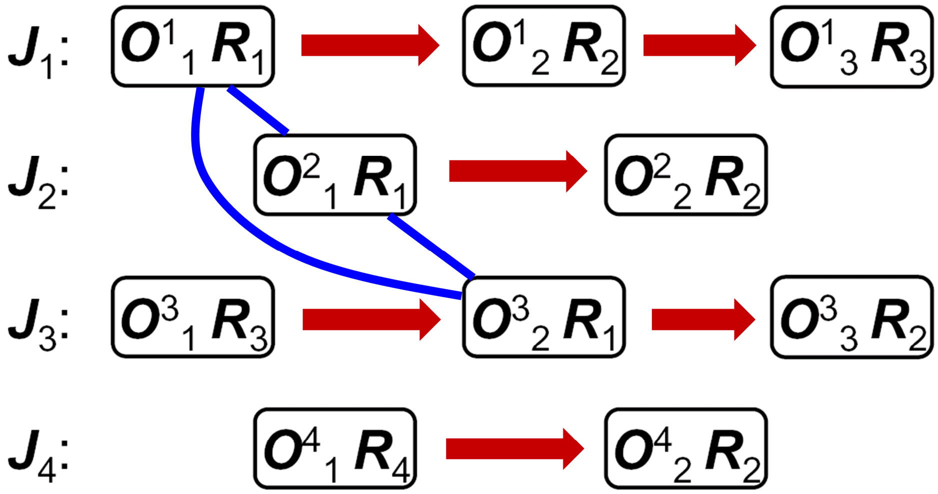
Precedence Constraints:

$S^1{}_1 + T^1{}_1 \leq S^1{}_2$
$S^1{}_2 + T^1{}_2 \leq S^1{}_3$
…

Deliver Constraints:

$S^1{}_3 + T^1{}_3 \leq \text{Due}$
$S^2{}_2 + T^2{}_2 \leq \text{Due}$
$S^3{}_3 + T^3{}_3 \leq \text{Due}$

$S^i{}_j$: start time for operation j of job i
$T^i{}_j$: duration of operation j of job i



J_1: $O^1_1\ R_1$ → $O^1_2\ R_2$ → $O^1_3\ R_3$

J_2: $O^2_1\ R_1$ → $O^2_2\ R_2$

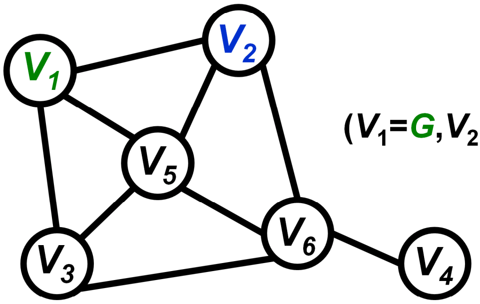
J_3: $O^3_1\ R_3$ → $O^3_2\ R_1$ → $O^3_3\ R_2$

J_4: $O^4_1\ R_4$ → $O^4_2\ R_2$

Resource constraints

Operations (1,1), (2,1), and (3,2) share the same resource R1

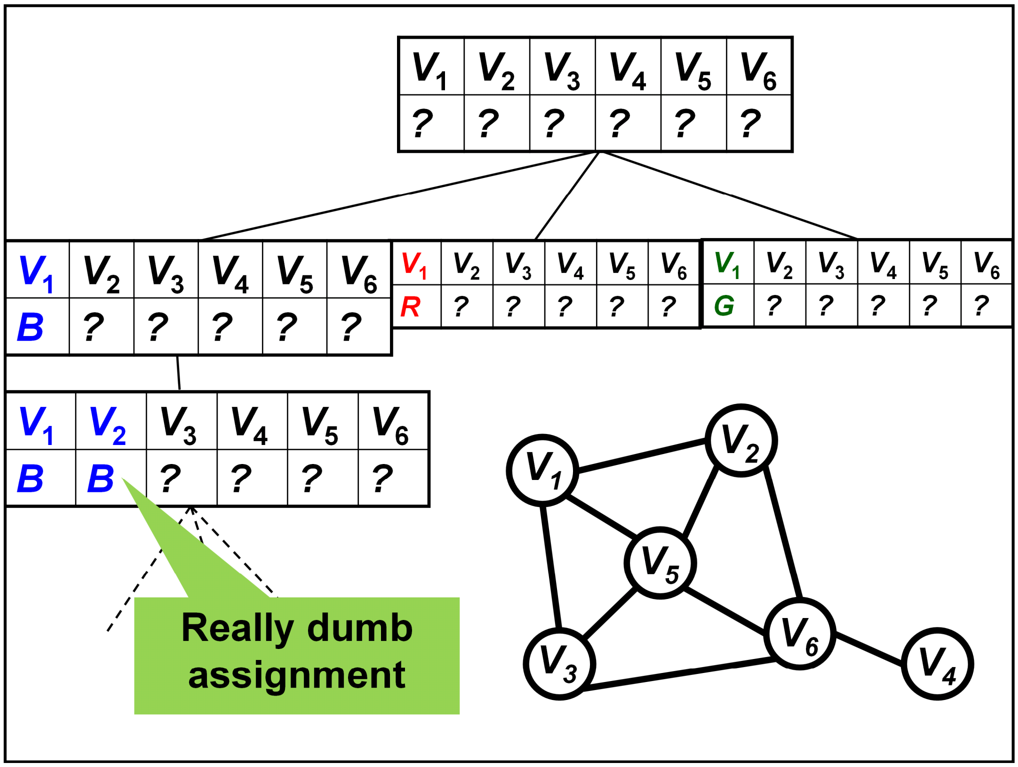
CSP as a Standard Search Problem



Example state:
(V_1=*G*,V_2=*B*, V_3=?, V_4=?, V_5=?, V_6=?)

- ***State*: assignment to *k* variables with *k*+1,..,*N* unassigned**
- ***Successor*: Assignment of a value to variable *k*+1, keeping the others unchanged**
- ***Start state*: (V_1=?,V_2=?, V_3=?, V_4=?, V_5=?, V_6=?)**
- ***Goal state*: All variables assigned with constraints satisfied**
- **No concept of cost on transition; just a solution, no path**

How to use DFS? For example, you'd have a tree with root as start state, children as V1=red, V1=blue, V1=green, and then branch out on V2, etc.



How many possible successors? Number of colors.

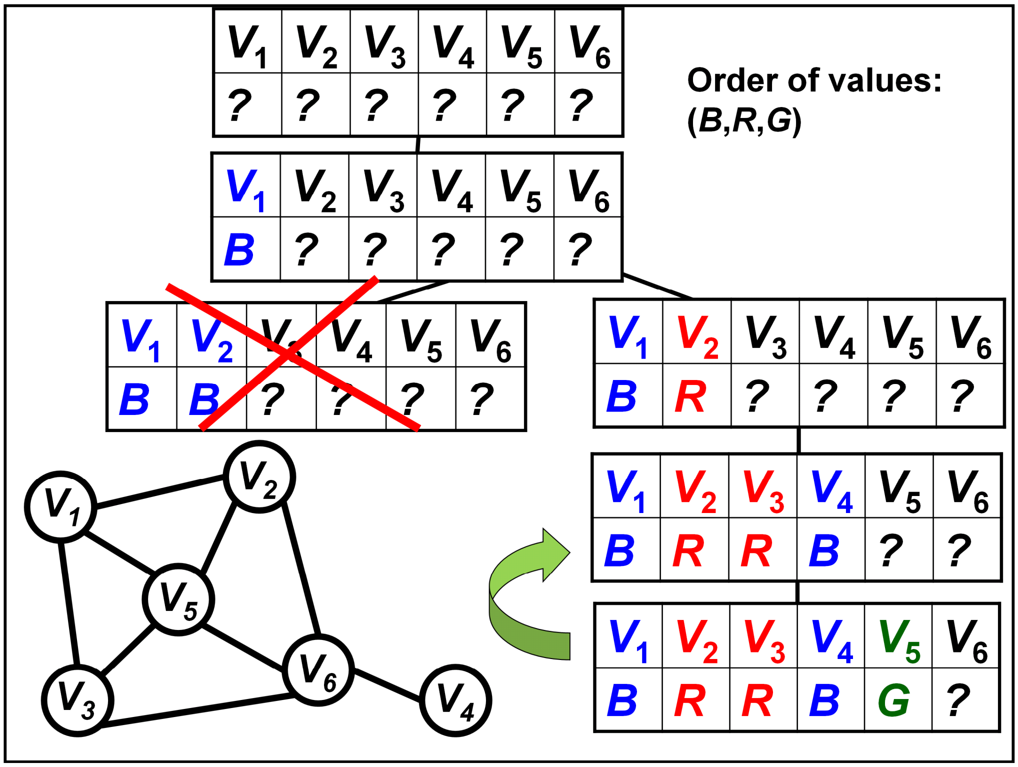
You'll get to the bottom of the tree and have to backtrack because you made a dumb assignment.

You can check if V1 and V2 don't violate.

How do you pick which node to explore next? Choose the most constrained node. But we're not going to get that smart yet.

DFS Improvements

- Evaluate only value assignments that do not violate any constraints with the current assignments
- Don’t search branches that obviously cannot lead to a solution
- Predict valid assignments ahead
- Control order of variables and values



Order of values:
(B,R,G)

V_1	V_2	V_3	V_4	V_5	V_6
?	?	?	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	?	?	?	?	?

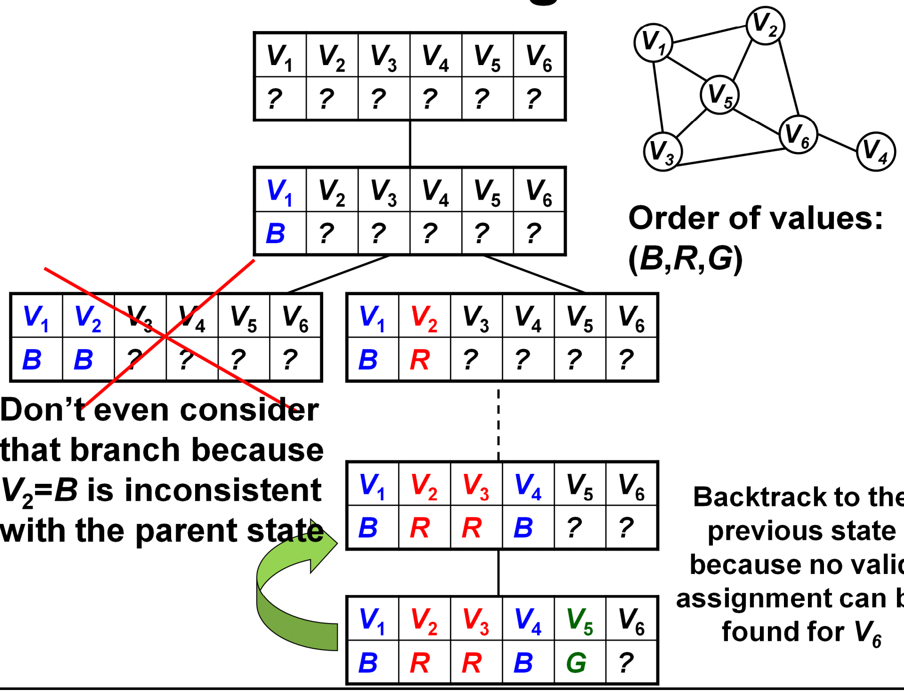
V_1	V_2	V_3	V_4	V_5	V_6
B	B	?	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	R	?	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	R	R	B	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	R	R	B	G	?

Backtracking DFS



V_1	V_2	V_3	V_4	V_5	V_6
?	?	?	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	?	?	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	*B*	?	?	?	?

Don't even consider that branch because V_2=*B* is inconsistent with the parent state

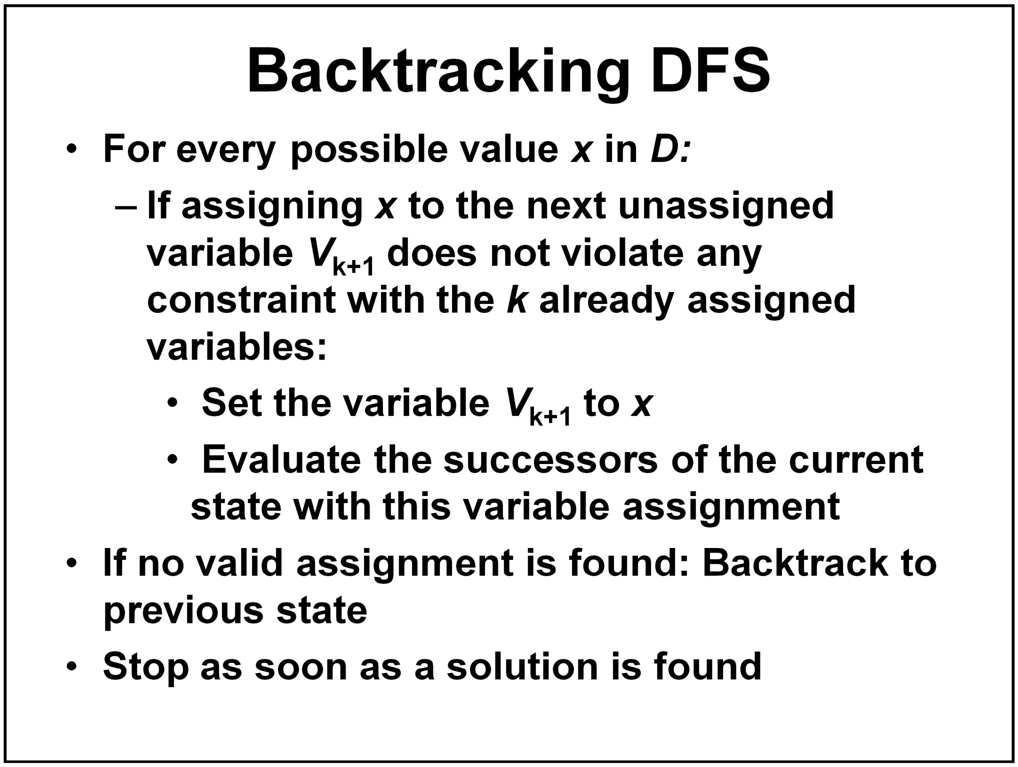
V_1	V_2	V_3	V_4	V_5	V_6
B	*R*	?	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	*R*	*R*	*B*	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	*R*	*R*	*B*	*G*	?

Order of values: (*B*,*R*,*G*)

Backtrack to the previous state because no valid assignment can be found for V_6



Backtracking DFS

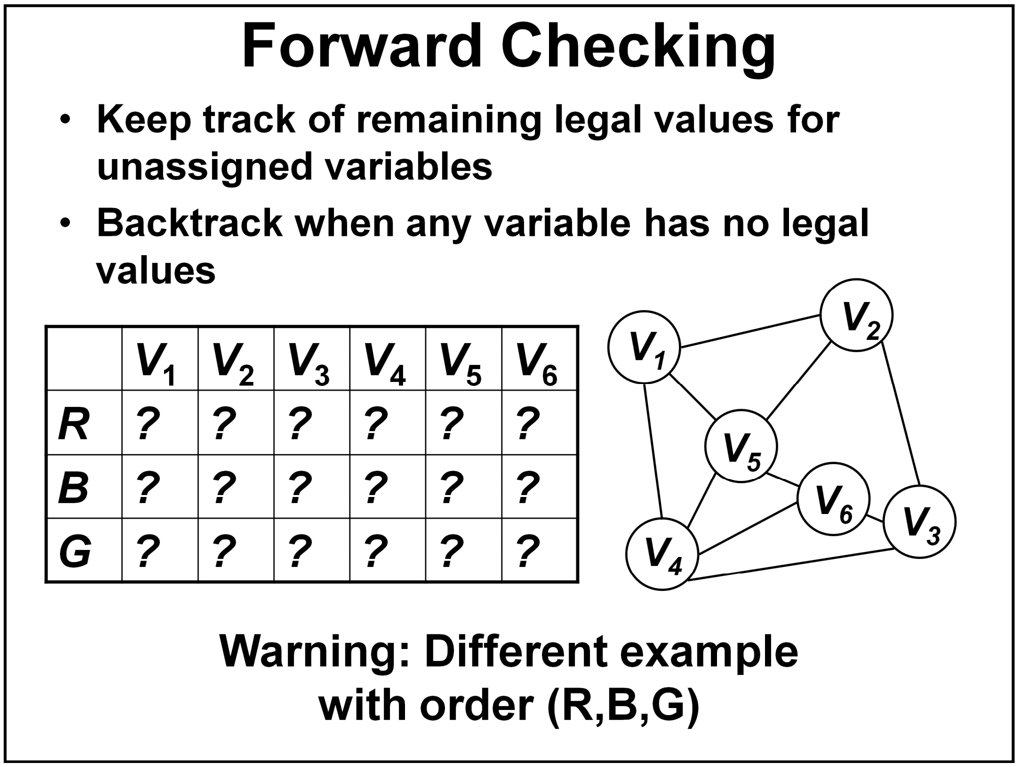
- For every possible value *x* in *D:*
  - If assigning *x* to the next unassigned variable V_{k+1} does not violate any constraint with the *k* already assigned variables:
    - Set the variable V_{k+1} to *x*
    - Evaluate the successors of the current state with this variable assignment
- If no valid assignment is found: Backtrack to previous state
- Stop as soon as a solution is found

This can be really slow.

Backtracking DFS Comments

- Additional computation: At each step, we need to evaluate the constraints associated with the current candidate assignment (variable, value).

- Uninformed search, we can improve by predicting:
  - What is the effect of assigning a variable on all of the other variables?
  - Which variable should be assigned next and in which order should the values be evaluated?
  - When a branch fails, how can we avoid repeating the same mistake?



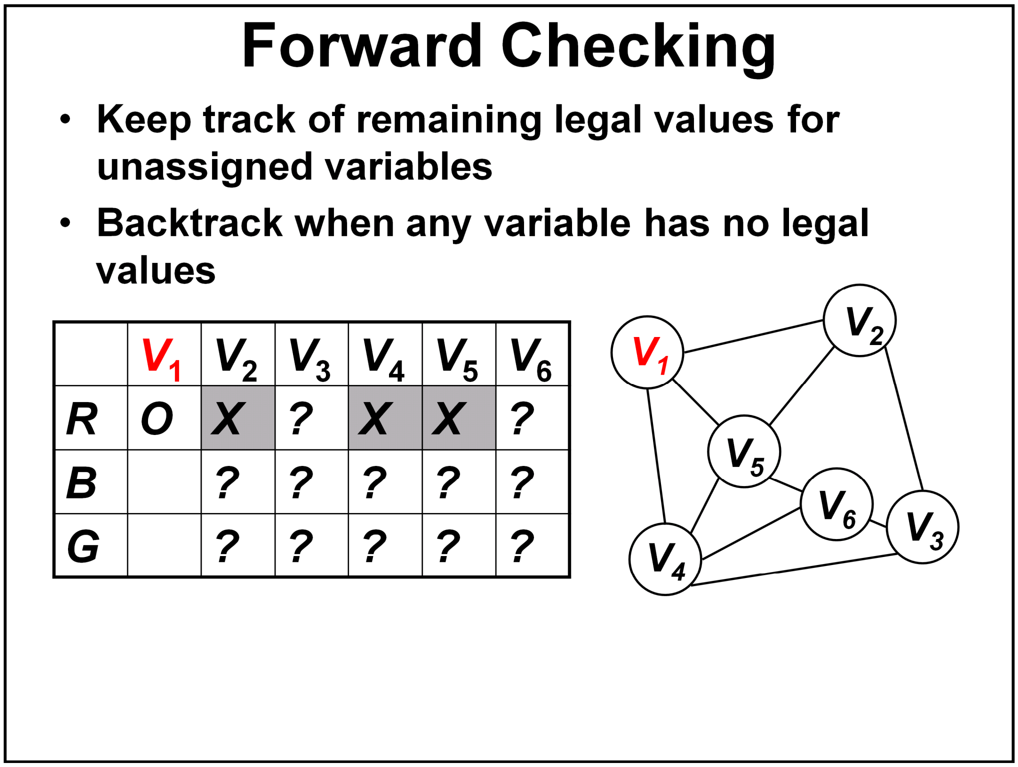
Forward Checking

- Keep track of remaining legal values for unassigned variables
- Backtrack when any variable has no legal values

	V_1	V_2	V_3	V_4	V_5	V_6
R	?	?	?	?	?	?
B	?	?	?	?	?	?
G	?	?	?	?	?	?

Warning: Different example with order (R,B,G)

This works slightly better than DFS.



Forward Checking

- **Keep track of remaining legal values for unassigned variables**
- **Backtrack when any variable has no legal values**

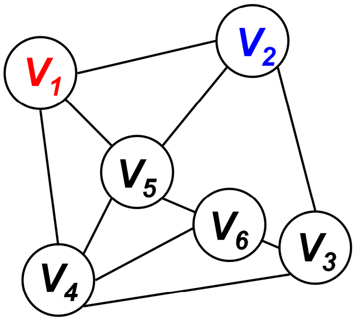
	V_1	V_2	V_3	V_4	V_5	V_6
R	O	X	?	X	X	?
B		?	?	?	?	?
G		?	?	?	?	?

Place X's in places that can't be red anymore. If V_1 is red, then V_2 V_4 and V_5 can't be assigned red.

Forward Checking

- Keep track of remaining legal values for unassigned variables
- Backtrack when any variable has no legal values

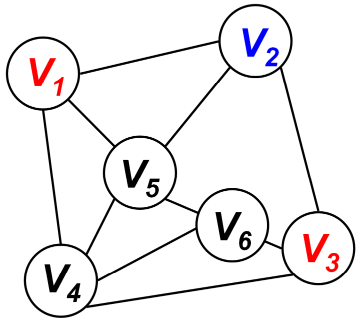
	V_1	V_2	V_3	V_4	V_5	V_6
R	O		?	X	X	?
B		O	X	?	X	?
G			?	?	?	?



Forward Checking

- **Keep track of remaining legal values for unassigned variables**
- **Backtrack when any variable has no legal values**

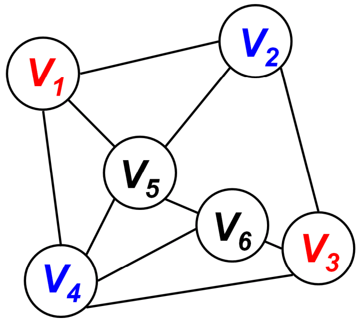
	V_1	V_2	V_3	V_4	V_5	V_6
R	O		O	X	X	X
B		O		?	X	?
G				?	?	?

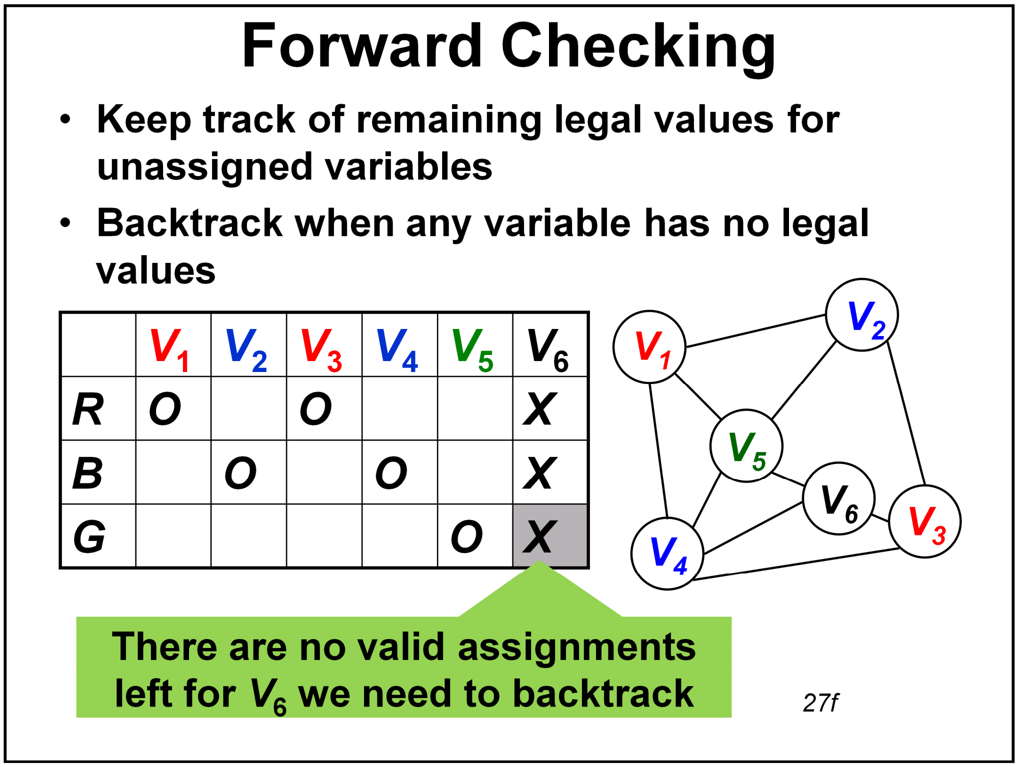


Forward Checking

- Keep track of remaining legal values for unassigned variables
- Backtrack when any variable has no legal values

	V_1	V_2	V_3	V_4	V_5	V_6
R	O		O		X	X
B		O		O	X	X
G					?	?





Forward Checking

- Keep track of remaining legal values for unassigned variables
- Backtrack when any variable has no legal values

	V_1	V_2	V_3	V_4	V_5	V_6
R	O		O			X
B		O		O		X
G					O	X

There are no valid assignments left for V_6 we need to backtrack

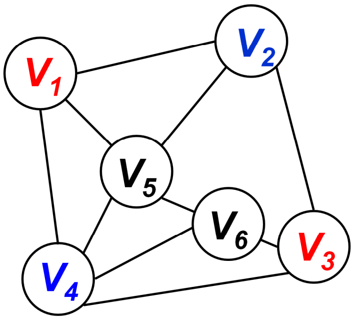
27f

The X’s mean there are whole branches of the search tree that we don’t look at. We still need to backtrack, but this is much more efficient. This is an example of pruning a search tree (chopping off branches)

Constraint Propagation

- Forward checking does not detect all the inconsistencies, only those that can be detected by looking at the constraints which contain the current variable.
- Can we look ahead further?

	V_1	V_2	V_3	V_4	V_5	V_6
R	*O*		*O*		*X*	*X*
B		*O*		*O*	*X*	*X*
G					*?*	*?*



At this point, it is already obvious that this branch will not lead to a solution because there are no consistent values in the remaining domain for V_5 and V_6.

Constraint Propagation,

not “just” checking

- **_V_ = variable being assigned at the current level**
- **Set variable _V_ to a value in _D_(_V_)**
- **For every variable _V’_ connected to _V_:**
  - **Remove the values in _D_(_V’_) that are inconsistent with the assigned variables**
  - **For every variable _V”_ connected to _V’_:**
    - **Remove the values in _D_(_V”_) that are no longer possible candidates**
    - **And do this again with the variables connected to _V”_**
      - **…until no more values can be discarded**

This is even better.

Constraint Propagation,

not “just” ch

Forward Checking as before

New: Constraint Propagation

- *V* = variable being assigne
  lue in *D*(
  connect to *V*:
  - F ove the values in *D*(*V’*) that are in sistent with the assigned variables
  - For very variable *V”* connected to *V’*:
    - Re nove the values in *D*(*V”*) that are no longer possible candidates
    - And do this again with the variables connected to *V”*
      - …until no more values can be discarded

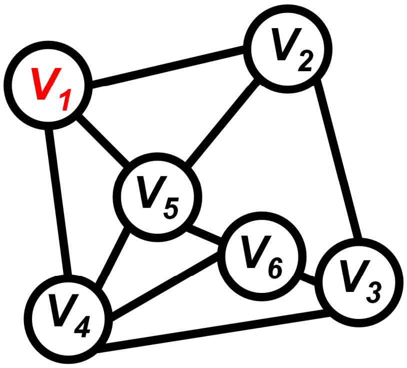
CP For Graph Coloring

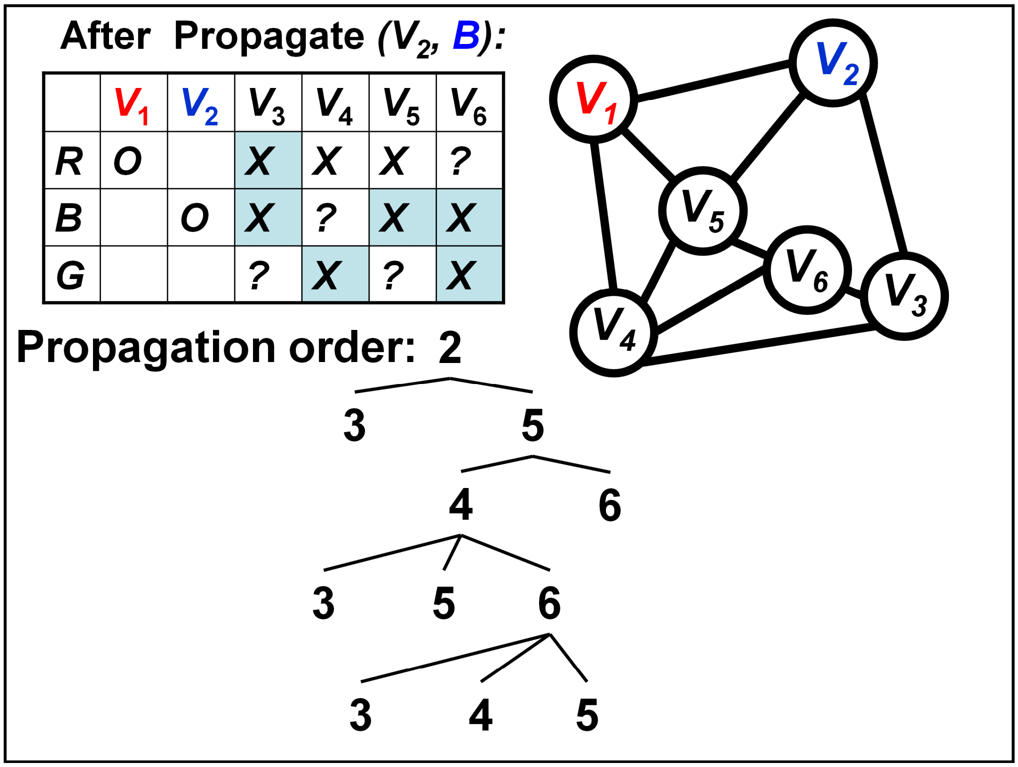
Propagate (*node*, *color*)

1. **Remove color from the domain of all of the neighbors**
2. **For every neighbor *N*:**
    - **If *D*(*N*) was reduced to only one color after step 1 (*D*(*N*) = {*c*}):**
    - **Propagate (*N*,*c*)**

After Propagate (V_1, R):

	V_1	V_2	V_3	V_4	V_5	V_6
R	O	X	?	X	X	?
B		?	?	?	?	?
G		?	?	?	?	?





After Propagate (V_2, B):

	V_1	V_2	V_3	V_4	V_5	V_6
R	O		X	X	X	?
B		O	X	?	X	X
G			?	X	?	X

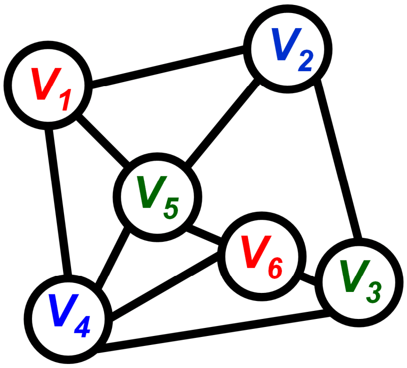
Propagation order: 2

Uh oh, V5 can only be one color. Take it. This is a key difference

Now that you set V_5, V_4 can now only be one color!

After Propagate *(V_2, B):*

	V_1	V_2	V_3	V_4	V_5	V_6
R	O		X	X	X	?
B		O	X	?	X	X
G			?	X	?	X



Note: We get directly to a solution in *one step of CP* after setting V_2 *without any additional search*

Some problems can even be solved by applying CP directly without search

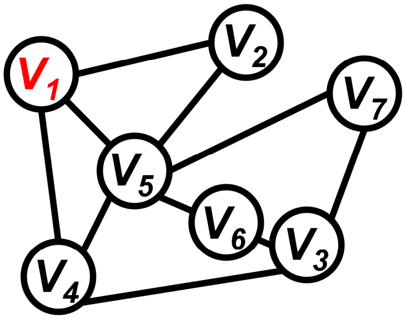
Variable and Value Heuristics

So far we have selected the next variable and the next value by using a fixed order

1. Is there a better way to pick the next variable?
2. Is there a better way to select the next value to assign to the current variable?

CSP Heuristics: Variable Ordering I

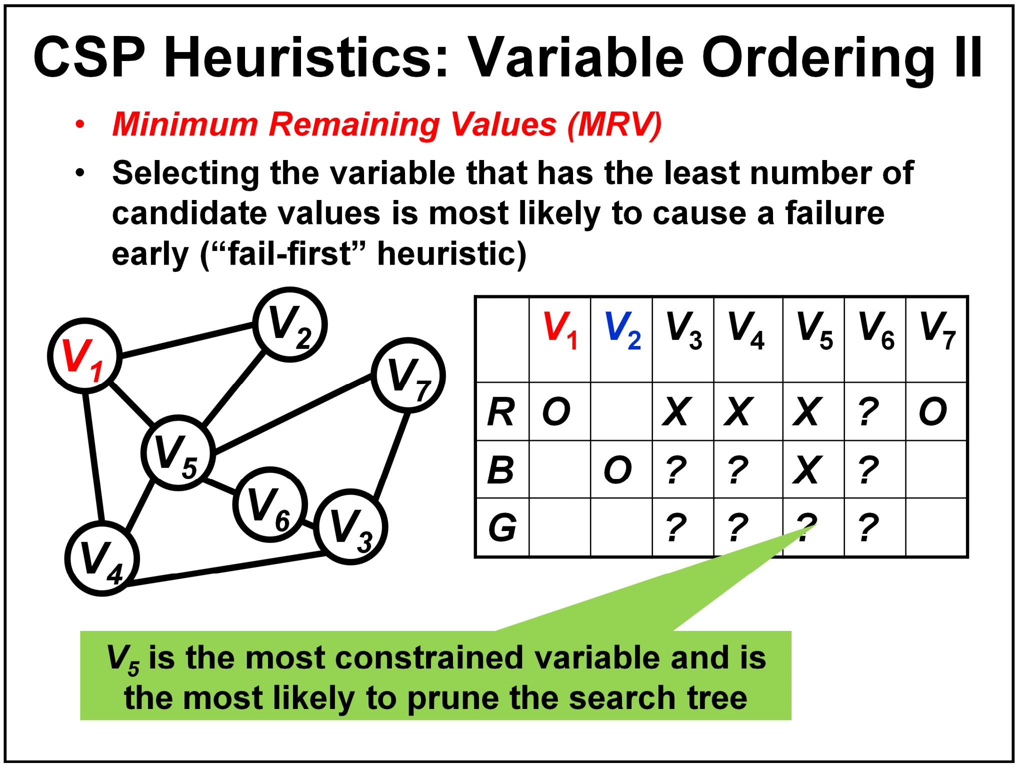
- ***Most Constraining Variable***
- **Selecting a variable which contributes to the *largest* number of constraints will have the largest effect on the other variables**
- **Equivalent to finding the variable that is connected to the largest number of variables in the constraint graph.**



V_1	V_2	V_3	V_4	V_5	V_6	V_7
R	*?*	*?*	*?*	*?*	*?*	*?*

Setting variable V_5 affects *4* variables

Setting variable V_2 (or V_3, V_4) affects fewer variables



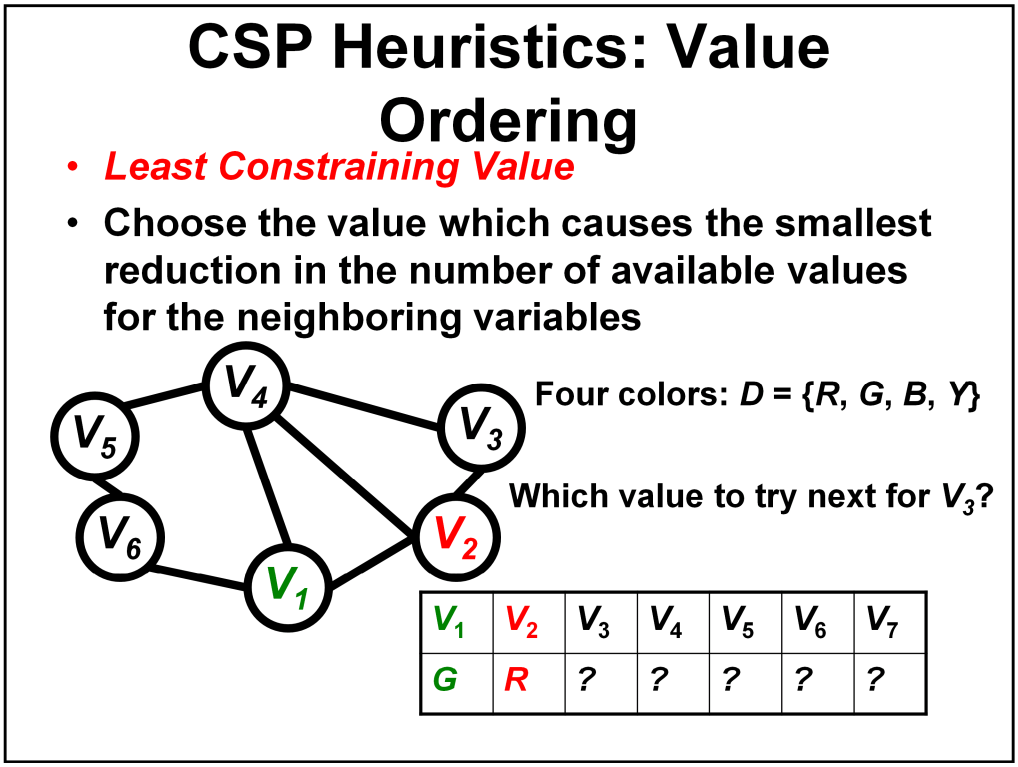
CSP Heuristics: Variable Ordering II

- *Minimum Remaining Values (MRV)*
- Selecting the variable that has the least number of candidate values is most likely to cause a failure early ("fail-first" heuristic)

	V_1	V_2	V_3	V_4	V_5	V_6	V_7
R	O		X	X	X	?	O
B		O	?	?	X	?	
G			?	?	?	?	

V_5 is the most constrained variable and is the most likely to prune the search tree

Fail-first is good, because you don't waste your time looking at branches of the search tree that can't possibly work.



CSP Heuristics: Value Ordering

- ***Least Constraining Value***
- **Choose the value which causes the smallest reduction in the number of available values for the neighboring variables**

Four colors: $D = \{R, G, B, Y\}$

Which value to try next for V_3?

V_1	V_2	V_3	V_4	V_5	V_6	V_7
G	*R*	*?*	*?*	*?*	*?*	*?*

Pick green for V_3 as it adds no new constraints (We already know that V_4 can’t be green!)

Conclusion – Generic CSP Solution

- Repeat until all variables have been assigned:
- Apply a consistency enforcement procedure
  - Forward checking
  - Constraint propagation
- If no solutions left:
  - Backtrack to a previous variable
- Else
  - select the next variable to be assigned
    - Using variable ordering heuristic
  - Select a value to try for this variable
    - Using value ordering heuristic