15-381: AI
Informed Search
Fall 2009

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Chapter 4, Russell and Norvig
Thanks to all past 381 instructors, and
http://www.cs.cmu.edu/~awm/tutorials

Carnegie Mellon
# Uninformed Search Complexity

- $N =$ Total number of states
- $B =$ Average number of successors (branching factor)
- $L =$ Length for start to goal with smallest number of steps
- $Q =$ Average size of the priority queue
- $L_{\text{max}} =$ Length of longest path from $\text{START}$ to any state

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Complete</th>
<th>Optimal</th>
<th>Time</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>BFS</td>
<td>Y</td>
<td>Y, If all trans. have same cost</td>
<td>$O(B)$</td>
<td>$O(B)$</td>
</tr>
<tr>
<td>BIBFS</td>
<td>Y</td>
<td>Y, If all trans. have same cost</td>
<td>$O(2B^{1/2})$</td>
<td>$O(2B^{1/2})$</td>
</tr>
<tr>
<td>PCDFS</td>
<td>Y</td>
<td>N</td>
<td>$O(B_{\text{max}})$</td>
<td>$O(BL_{\text{max}})$</td>
</tr>
<tr>
<td>MeMУ+</td>
<td>Y</td>
<td>N</td>
<td>$O(B_{\text{max}})$</td>
<td>$O(BL_{\text{max}})$</td>
</tr>
<tr>
<td>IDS</td>
<td>Y</td>
<td>Y, If all trans. have same cost</td>
<td>$O(B)$</td>
<td>$O(BL)$</td>
</tr>
</tbody>
</table>

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15-381 AI  
Fall 09
How does DFS expand? How do you decide which next node? Use a stack…expand the states most recently added to the stack.

How does BFS expand? Use a queue…expand the states that were added first to the queue.

Note: In these examples s is just a state, not necessarily a start state

1. Define some function \( f(s) \) at each state \( s \)
2. Choose the state with “lowest” \( f \) to expand next
3. Insert its successors

If \( f() \) is chosen carefully, we will eventually find the lowest-cost sequence
Uninformed vs Informed

- Uninformed – only guided by
  - *successor* relationships
  - topological structure (leftmost,...)
  - length as number of nodes

- Informed
  - assume *cost* of edges
  - more knowledge?
Uninformed: has no information about how to get to the goal!

- UCS (Uniform Cost Search) \( f(n) = g(n) \)

- **g(n)** - cost of each node already expanded
  \( \text{length of shortest path from START to } n \)

- Implementation – Store open successor states (waiting to be expanded) in a *priority queue* for efficient retrieval of minimum \( f \)

- Optimal \( \Rightarrow \) Guaranteed to find lowest cost sequence, *but guidance is about known path*…

Uninformed: has no information about how to get to the goal!
We really want a “best guess”– as informed as possible.
Informed: Concept of “external knowledge”. Use more than just the state and the actions.

For those of you who have taken 15-211: You wrote a heuristic for your chess playing AI. Your heuristic would take the state (a state of the chess board) and output how good or bad the board is, which is an estimate of how close you are to winning.
You can only move North, South, East, or West. Manhattan distance is the number of moves you must make when moving this way on a grid. In this example, the Manhattan distance from S to the goal is 8 (4 North, 4 East).

Is Euclidean distance an accurate estimate?
No- because you move based on Manhattan distance.
Is the Euclidean distance ever greater than the Manhattan distance? No. Euclidean is a *lower bound*. It always underestimates the distance you must travel, since you can’t go in a straight line or pass through walls.
Simon got the Nobel for introducing this concept of heuristic.
A guess of what’s best, but not a proven best.
Heuristic Functions Example

- $h(s) = $ Euclidean distance to GOAL
- Euclidean distance is an heuristic.
Heuristic Functions Example

- How could we define $h(s)$?
Why is $h_2$ better?

See Pearl- even more sophisticated heuristics.

**Misplaced titles:**

$h_1(s) = 7$

**Manhattan distance:**

$h_2(s) = 2 + 3 + 3 + 2 + 4 + 2 + 0 + 2 = 18$
How is this different from UCS?

Heuristic value of state is a property of state, not the path to get to the state. (All step costs are the same).

\( h(s) \) is independent of path to reach \( s \)

BFS is generally good, but there can be some problems…
Queue for Greedy BFS is shown, and we always take the lowest $h$ value. Path is first node in each step if sorted by best heuristic.

But this isn’t the shortest path in terms of cost. So GBFS not optimal, even though heuristic function is “good”. $H$ always *underestimates* shortest path to goal.

But GBFS is very easy to implement, and there are problems for which it is great.
Can also do a modification— put weights:

\[ F(n) = \alpha \cdot g(n) + (1-\alpha) \cdot h(n). \]

You can play around with the alphas.
Heuristics should be easy to define—we don’t want heuristics to take longer than the search itself.

Defining good heuristics is very important and can be very complicated. For a search problem, choosing the heuristic is a big decision.
Can A* Fix the Problem?

\[
\begin{align*}
\text{START} & \quad h = 4 \\
A & \quad h = 3 \\
B & \quad h = 2 \\
C & \quad h = 1 \\
\text{GOAL} & \quad h = 0 \\
\end{align*}
\]

\[
\{(\text{START},4)\} \\
\{(A,5)\} \\
\{(B,5)\} \\
\{(C,7)\} \\
\{(C,5)\} \\
\{(\text{GOAL},6)\}
\]

\[
\begin{align*}
(f(A)) &= g(A) + h(A) = g(\text{START}) + \text{cost}(\text{START},A) + 3 = 0 + 2 + 3 \\
\{(B,5)\} \\
\{(C,7)\} \\
\end{align*}
\]

\[
\begin{align*}
(f(C)) &= g(C) + h(C) - g(A) + \text{cost}(A,C) + 1 - 2 + 4 + 1 \\
\{(C,5)\} \\
\end{align*}
\]

\[
\begin{align*}
(f(C)) &= g(C) + h(C) = g(B) + \text{cost}(B,C) + 1 = 3 + 1 + 1 \\
\{(\text{GOAL},6)\} \\
\end{align*}
\]
In implementation you have to keep track of both $g$ and $h$ throughout.

Be disciplined running A* and don’t take intuitive shortcuts. Manuela is “a master” at coming up with graphs designed to trick you. And she’s been teaching this since 1992. That’s, like, a really long time!
These are what’s really important.

A* Core Issues

- Termination condition
- Revisiting states
- Algorithm
- Optimality
- Avoiding revisiting states
- Choosing good heuristics
- Reducing memory usage
If we stopped as soon as we see G, we’d lose. Would we stop if we had (G,8)? Yes. And you wouldn’t find the optimal path because your heuristic sucks. Same if \( h(A) = 20 \). You’d pop (G,10) and you’d stop and lose. However, Manuela will blame the bad heuristic for overestimating the cost to the goal from A.

Bad is well-defined too, not just slang.
Unlike in GBFS, you have to revisit states.

- \((\text{Start}, 8)\)
- \((\text{B}, 4), (\text{A}, 8)\)
- \((\text{A}, 8), (\text{C}, 10)\)
- \((\text{C}, 9.5)\) (C has been updated!)
- \((\text{D}, 3.5)\)
- \((\text{Goal}, 9.5)\)

Done! (Popped the goal)
Two cases:

1) Revisited state \( s \) is still in PQ. If new \( g(s) \) is smaller than old, update it.

2) Revisited state \( s \) has already been expanded. If new \( g(s) \) is smaller than old, re-insert it

(Start, 8)
(B, 4), (A, 8)
(C, 4), (A, 8)
(D, 4), (A, 8)
(A, 8), (Goal, 10)
(C, 3.5), (Goal, 10) (Re-inserted C into PQ!)
(D, 3.5), (Goal, 10) (Re-inserted D into PQ!)
(Goal, 9.5) (Updated Goal)

Done! (popped goal)

Pay attention to these two slides, walk through it yourself.
Pop state $s$ with lowest $f(s)$ in queue
If $s = GOAL$
    return $SUCCESS$
Else expand $s$:
For all $s'$ in $\text{successes}(s)$:
    $f' = g(s') + h(s') = g(s) + \text{cost}(s, s') + h(s')$
    If ($s'$ not seen before OR
        $s'$ previously expanded with $f(s') > f'$ OR
        $s'$ in $PQ$ with with $f(s') > f$)
        Promote/Insert $s'$ with new value $f'$ in $PQ$
        $\text{previous}(s') \leftarrow s$
    Else
        Ignore $s'$ (because it has been visited and
        its current path cost $f(s')$ is still the lowest
        path cost from $START$ to $s'$)
Why doesn’t it find the optimal?  H is *overestimating*.

A good heuristic does not overestimate.

Let \( h^* \) be actual cost from \( n \) to goal.

\[ h(n) \leq h^*(n) \]

We’ve done one uninformed search that was always optimal (BFS).

What was the heuristic for BFS?  \( h(s) = c \), where \( c \) is some constant.  Therefore 
\[ f(s) = g(s) + h(s) = g(s) + c. \]
Admissible Heuristics

- Define $h^*(s) = \text{the true minimal cost to the goal from } s$
- $h$ is admissible if
  \[ h(s) \leq h^*(s) \text{ for all states } s \]
- I.e., an admissible heuristic never overestimates the cost to the goal.
  “Optimistic” estimate of cost to goal.

A* is guaranteed to find the optimal path if $h$ is admissible.

To be continued next lecture! Stay tuned, true believers!
Consistent (Monotonic) Heuristics

\[ h(s) \leq h(s') + \text{cost}(s, s'), \; h(G) = 0 \]

Triangular inequality implies that path cost always increases + need to expand node only once

f values are monotonically nondecreasing, \( f(s') \geq f(s) \)
Pop state \( s \) with lowest \( f(s) \) in queue

If \( s = GOAL \)
   return \( SUCCESS \)
Else expand \( s \):
   For all \( s' \) in \( \text{succs}(s) \):
      \[
      f' = g(s') + h(s') = g(s) + \text{cost}(s, s') + h(s')
      \]
      If \( (s' \) not seen before OR \( s' \) previously expanded with \( f(s') > f' \) OR \( s' \) in PQ with with \( f(s') > f \) )
         Promote/Insert \( s' \) with new value \( f' \) in PQ
      \[
      \text{previous}(s') \leftarrow s
      \]
   Else
      Ignore \( s' \) (because it has been visited and its current path cost \( f(s') \) is still the lowest path cost from \( \text{START} \) to \( s' \))
Examples

For the navigation problem:
The length of the shortest path is at least the distance between \( s \) and GOAL →
Euclidean distance is an admissible heuristic

What about the puzzle?

\[
\begin{bmatrix}
1 & 5 & 3 \\
8 & 2 & 4 \\
7 & 6 & \\
\end{bmatrix}
\quad \text{h(s) ?} \quad \begin{bmatrix}
1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8 \\
\end{bmatrix}
\]
Are these heuristics admissible?
They both underestimate the number of moves you need to solve!

Misplaced titles:
\[ h_1(s) = 7 \]

Manhattan distance:
\[ h_2(s) = 2 + 3 + 3 + 2 + 4 + 2 + 0 + 2 = 18 \]
Comparing Heuristics – States expanded

<table>
<thead>
<tr>
<th></th>
<th>$L = 4$ steps</th>
<th>$L = 8$ steps</th>
<th>$L = 12$ steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iterative Deepening</td>
<td>112</td>
<td>6,384</td>
<td>364,404</td>
</tr>
<tr>
<td>A* with heuristic $h_1$</td>
<td>13</td>
<td>39</td>
<td>227</td>
</tr>
<tr>
<td>A* with heuristic $h_2$</td>
<td>12</td>
<td>25</td>
<td>73</td>
</tr>
</tbody>
</table>

- Data is averaged over 100 instances of the 8-puzzle for various solution lengths.

Using a heuristic you only have to look at far, far fewer nodes!
Manhattan distance $h_2(s)$ is a tighter lower bound, so it works better.

Domination is always good if the dominating heuristic is admissible, however domination does NOT imply admissibility.
Limitations

- Computation: In the worst case, we may have to explore all the states

- The good news: A* is optimally efficient → For a given $h(\cdot)$, no other optimal algorithm will expand fewer nodes
IDS (Iterative Deepening Search)

- Need to make DFS optimal

- IDS (Iterative Deepening Search):
  - Run DFS by searching only path of length 1
    (DFS stops if length of path is greater than 1)
  - If that doesn’t find a solution, try again by
    running DFS on paths of length 2 or less
  - If that doesn’t find a solution, try again by
    running DFS on paths of length 3 or less
  - ..........
  - Continue until a solution is found
Example: IDA* (Iterative Deepening A*)

- Same idea as Iterative Deepening DFS except use $f(s)$ to control depth of search instead of the number of transitions
- Example, assuming integer costs:
  
  1. Run DFS, stopping at states $s$ such that $f(s) > 0$
     Stop if goal reached
  2. Run DFS, stopping at states $s$ such that $f(s) > 1$
     Stop if goal reached
  3. Run DFS, stopping at states $s$ such that $f(s) > 2$
     Stop if goal reached
  
  ........Keep going by increasing the limit on $f$ by 1 every time

- Complete
- Optimal
- More expensive in computation cost than A*
- Memory order $A$ as in DFS

For IDS you iterate depth.
For IDA* you iterate $f(s)$
Summary

- Informed search and heuristics
- Best-First Greedy search
- A* algorithm
  - Admissible heuristics, optimality
  - Condition on heuristic functions
  - Completeness, efficiency
- IDA*

Chapters 3&4 Russell & Norvig
Proof of A* Optimality with Admissible h

- By contradiction – assume that a suboptimal goal state, G’ is returned.
- Let G be a goal state with optimal path cost f* and let n be a node in the path to G. h admissible, therefore f* >= f(n)
- If n is missed for expansion and instead G’ is chosen, then f(n) >= f(G’)
- So f* >= f(G’), and f* >= g(G’) + h(G’), and f* >= g(G’), which contradicts the assumption that G’ is suboptimal.

You don’t need to know this proof...just here in case you’re interested.