Symbolic Integration

Victor Adamchik

Carnegie Mellon University

Integrals of a Logarithmic Extension

Evaluate this integral

$$\int\left(x \log^2(x)+\frac{\log(x+1)}{(x+1)^2}+\frac{1}{x\log(x)}\right)dx$$

Solution.

Do not split integral into three. Such splitting is always dangerous. See the following integral

$$\int\left(-\frac{\left(1+\frac{1}{x}\right)\text{Log}[x]^2}{(x+\text{Log}[x])^2}+\frac{2\,\text{Log}[x]}{x\,(x+\text{Log}[x])}\right)dx$$

Extending Q[x] by $t_1 = \log x$ and $t_2 = \log(x+1)$, we obtain

$$x\,t_1^2+\frac{1}{x\,t_1}+\frac{t_2}{(x+1)^2}$$

Since is a polynomial wrt t_2, we assume that

$$\int\left(x\,t_1^2+\frac{1}{x\,t_1}+\frac{t_2}{(x+1)^2}\right)dx = b_2\,t_2{}^2+b_1\,t_2+b_0+\sum_{k=1}^{n} d_k * \log(u_k)$$

Differentiating both sides wrt x

$$x\,t_1^2+\frac{1}{x\,t_1}+\frac{t_2}{(x+1)^2} = b_2{}'\,t_2{}^2+(2\,b_2\,t_2{}'+b_1{}')\,t_2+b_1\,t_2{}'+b_0{}'+\frac{d}{dx}\sum_{k=1}^{n} d_k * \log(u_k)$$

and equating coefficients by t_2,we get a system of equations

$$\begin{aligned} b_2{}' &= 0 \\ 2\,b_2\,t_2{}'+b_1{}' &= \tfrac{1}{(1+x)^2} \\ b_1\,t_2{}'+b_0{}'+\tfrac{d}{dx}\textstyle\sum_{k=1}^{n} d_k * \log(u_k) &= x\,t_1^2+\tfrac{1}{x\,t_1} \end{aligned}$$

From the first equation we find

$$b_2 = c_2$$

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$$b_1{}' = -2\, c_2\, t_2{}' + \frac{1}{(1+x)^2}$$

$$b_1 = -2\, c_2\, t_2 - \frac{1}{1+x} + c_1$$

and conclude that

$$c_2 = 0$$

On this step we call integration in $Q[x]$

$$\int \frac{dx}{(x+1)^2} = -\frac{1}{1+x}$$

The last equation in the above system is

$$b_1\, t_2{}' + b_0{}' + \frac{d}{d\,x} \sum_{k=1}^{n} d_k * \log(u_k) = x\, t_1^2 + \frac{1}{x\, t_1}$$

$$\left(c_1 - \frac{1}{1+x}\right) t_2{}' + b_0{}' + \frac{d}{d\,x} \sum_{k=1}^{n} d_k * \log(u_k) = x\, t_1^2 + \frac{1}{x\, t_1}$$

$$b_0{}' + \frac{d}{d\,x} \sum_{k=1}^{n} d_k * \log(u_k) = x\, t_1^2 + \frac{1}{x\, t_1} - \left(c_1 - \frac{1}{1+x}\right) t_2{}'$$

$$b_0 + \sum_{k=1}^{n} d_k * \log(u_k) = \int \left(x\, t_1^2 + \frac{1}{x\, t_1} - \left(c_1 - \frac{1}{1+x}\right) \frac{1}{1+x}\right) \tag{1}$$

Here we again have a call to integration but this time it is in $Q[x,\ t_1]$. We break the integral in two - polynomial and rational parts

$$\int \left(x\, t_1^2 + \frac{1}{x\, t_1} - \left(c_1 - \frac{1}{1+x}\right) \frac{1}{1+x}\right) = \int \frac{1}{x\, t_1} + \int \left(x\, t_1^2 - \left(c_1 - \frac{1}{1+x}\right) \frac{1}{1+x}\right)$$

and integrate each of them separately.

Rational part

$$\int \frac{1}{x\, t_1}$$

Compute the resultant

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D[x t1[x], x]
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t1[x] + x t1'[x]
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Resultant[x t1, 1 - z (t1 + 1), t1]
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x (1 - z)
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and then compute GCD

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PolynomialGCD[x t1, 1 - 1 (t1 + 1)]
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t1
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Thus, we found

$$\int \frac{1}{x\, t_1} = 1 * \log t_1 + c$$

Polynomial part

$$\int \left(x\, t_1^2 - \left(c_1 - \frac{1}{1+x}\right) \frac{1}{1+x} \right)$$

We proceed in the same as we did above for t_2

Since is a polynomial wrt t_1, we assume that

$$\int \left(x\, t_1^2 - \left(c_1 - \frac{1}{1+x}\right) \frac{1}{1+x} \right) dx = b_3\, t_1{}^3 + b_2\, t_1{}^2 + b_1\, t_1 + b_0 + \sum_{k=1}^{n} d_k * \log(u_k)$$

Differentiating both sides wrt x

$$x\, t_1^2 - \left(c_1 - \frac{1}{1+x}\right) \frac{1}{1+x} =$$

$$b_2'\, t_1{}^2 + (3\, b_3\, t_1' + b_2')\, t_1{}^2 + (2\, b_2\, t_1' + b_1')\, t_1 + b_1\, t_1' + b_0' + \frac{d}{dx} \sum_{k=1}^{n} d_k * \log(u_k)$$

and equating coefficients by t_1,we get a system of equations

$$\begin{array}{cl}
b_3' & = 0 \\
3\, b_3\, t_1{}' + b_2' & = x \\
2\, b_2\, t' + b_1' & = 0 \\
b_1\, t_1' + b_0' + \frac{d}{dx} \sum_{k=1}^{n} d_k * \log(u_k) & = -\left(c_1 - \frac{1}{1+x}\right) \frac{1}{1+x}
\end{array}$$

From the first equation we find that b_3 of x

$$b_3 = C_3$$

second equation

$$3\, C_3\, t_1 + b_2 = \frac{x^2}{2} + C_2$$

$$C_3 = 0 \quad \text{and} \quad b_2 = \frac{x^2}{2} + C_2$$

third equation

$$2\, b_2\, t' + b_1{}' = 0$$

$$2\left(\frac{x^2}{2} + C_2\right) t_1{}' + b_1{}' = 0$$

$$b_1{}' = -2\left(\frac{x^2}{2} + C_2\right)\frac{1}{x}$$

$$b_1 = -\frac{x^2}{2} - 2\, t_1\, C_2 \; + \; C_1$$

$$C_2 = 0 \quad \text{and} \quad b_1 = -\frac{x^2}{2} + C_1$$

forth equation

$$b_1\, t_1{}' + b_0{}' + \frac{d}{d\,x}\sum_{k=1}^{n} d_k * \log\left(u_k\right) = -\left(c_1 - \frac{1}{1+x}\right)\frac{1}{1+x}$$

$$b_0{}' + \frac{d}{d\,x}\sum_{k=1}^{n} d_k * \log\left(u_k\right) = -\left(c_1 - \frac{1}{1+x}\right)\frac{1}{1+x} - \left(-\frac{x^2}{2} + C_1\right)\frac{1}{x}$$

$$b_0 + \sum_{k=1}^{n} d_k * \log\left(u_k\right) = \int\left[-\left(c_1 - \frac{1}{1+x}\right)\frac{1}{1+x} - \left(-\frac{x^2}{2} + C_1\right)\frac{1}{x}\right] + C_0$$

$$b_0 + \sum_{k=1}^{n} d_k * \log\left(u_k\right) = \frac{x^2}{4} - \log(x+1)\, c_1 - \log(x)\, C_1 - \frac{1}{x+1} + C_0$$

Remember, on the last step we do allow a field extension! It follows

$$b_0 = \frac{x^2}{4} - \frac{1}{x+1} + C_0$$

$$d_1 * \log\left(u_1\right) + d_2 * \log\left(u_2\right) = -\log(x+1)\, c_1 - \log(x)\, C_1$$

So, we found that

$$\int \left(x\, t_1^2 - \left(c_1 - \frac{1}{1+x} \right) \frac{1}{1+x} \right) dx =$$

$$\frac{x^2}{2}\, t_1{}^2 + \left(-\frac{x^2}{2} + C_1 \right) t_1 + \left(\frac{x^2}{4} - \frac{1}{x+1} + C_0 \right) - \log(x+1)\, c_1 - \log(x)\, C_1 =$$

$$\frac{x^2}{2}\, t_1{}^2 - \frac{x^2}{2}\, t_1 + \left(\frac{x^2}{4} - \frac{1}{x+1} + C_0 \right) - \log(x+1)\, c_1$$

Finally, we combine polynomial and rational parts,

$$\int \left(x\, t_1^2 + \frac{1}{x\, t_1} - \left(c_1 - \frac{1}{1+x} \right) \frac{1}{1+x} \right) = \log t_1 + \frac{x^2}{2}\, t_1{}^2 - \frac{x^2}{2}\, t_1 + \left(\frac{x^2}{4} - \frac{1}{x+1} + C_0 \right) - \log(x+1)\, c_1$$

Back to equation (1)

$$b_0 + \sum_{k=1}^{n} d_k * \log(u_k) = \int \left(x\, t_1^2 + \frac{1}{x\, t_1} - \left(c_1 - \frac{1}{1+x} \right) \frac{1}{1+x} \right)$$

that is

$$b_0 + \sum_{k=1}^{n} d_k * \log(u_k) = \frac{x^2}{2}\, t_1{}^2 - \frac{x^2}{2}\, t_1 + \left(\frac{x^2}{4} - \frac{1}{x+1} + C_0 \right) - \log(x+1)\, c_1 + \log t_1$$

Therefore,

$$b_0 = \frac{x^2}{2}\, t_1{}^2 - \frac{x^2}{2}\, t_1 + \left(\frac{x^2}{4} - \frac{1}{x+1} + C_0 \right)$$

$$d_1 = 1; \;\; u_1 = t_1$$

$$d_2 = -c_1; \;\; u_2 = 1 + x$$

Putting all these together

$$\int \left(x\, t_1^2 + \frac{1}{x\, t_1} + \frac{t_2}{(x+1)^2} \right) dx = b_2\, t_2{}^2 + b_1\, t_2 + b_0 + \sum_{k=1}^{n} d_k * \log(u_k)$$

$$\int \left(x\, t_1^2 + \frac{1}{x\, t_1} + \frac{t_2}{(x+1)^2} \right) dx =$$

$$\left(c_1 - \frac{1}{1+x} \right) t_2 + \frac{x^2}{2}\, t_1{}^2 - \frac{x^2}{2}\, t_1 + \left(\frac{x^2}{4} - \frac{1}{x+1} + C_0 \right) - \log(x+1)\, c_1 + \log t_1$$

or

$$\int \left(x \log^2(x) + \frac{\log(x+1)}{(x+1)^2} + \frac{1}{x \log(x)} \right) d\,x =$$

$$\frac{x^2}{2} \log^2 x - \frac{x^2}{2} \log x - \frac{\log(x+1)}{x+1} + \log\log x + \left(\frac{x^2}{4} - \frac{1}{x+1} \right)$$