Symbolic Integration

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Introduction

Several years ago, I was invited to contemplate being marooned
on the proverbial desert island. What book would I most wish to have there,
in addition to the Bible and the complete works of Shakespeare?
My immediate answer was Abramowitz and Stegun's
Handbook of Mathematical Functions.
If I could substitute for the Bible, I would choose Gradsteyn and Ryzhik' s
Table of Integrals, Series and Products.
Compounding the impiety, I would give up Shakespeare in favor of
Prudnikov, Brychkov and Marichev's *Integrals and Series*. - Michael Berry [1]

■ Calculus Integration

Consider a rational function

$$f = \frac{x^7 - 15\,x^5 - 7\,x^3 + 6\,x - 7}{x^5 - 6\,x^4 + 13\,x^3 - 12\,x^2 + 4\,x}$$

and compute its integral with the *Calculus* (or *Analysis*) method. By factoring the denominator

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Factor[x^5 - 6 x^4 + 13 x^3 - 12 x^2 + 4 x]
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(-2 + x)^2 (-1 + x)^2 x
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Thus, the original function f can be rewritten as

$$f = \frac{x^7 - 15\,x^5 - 7\,x^3 + 6\,x - 7}{x\,(x-1)^2\,(x-2)^2}$$

Next we rewrite f as a sum of terms with minimal denominators.

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Apart[(x^7 - 15 x^5 - 7 x^3 + 6 x - 7)/(x (x - 1)^2 (x - 2)^2)]
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$$8 - \frac{403}{2\,(-2+x)^2} + \frac{355}{4\,(-2+x)} - \frac{22}{(-1+x)^2} - \frac{105}{-1+x} - \frac{7}{4\,x} + 6\,x + x^2$$

Thus

$$f\; = 8 + 6\,x + x^2 + -\frac{7}{4\,x} - \frac{22}{(x-1)^2} - \frac{105}{x-1} - \frac{403}{2\,(x-2)^2} + \frac{355}{4\,(x-2)}$$

This can be easily integrated term by term

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Integrate[#, x] & /@ %
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$$\frac{403}{2\,(-2+x)} + \frac{22}{-1+x} + 8\,x + 3\,x^2 + \frac{x^3}{3} + \frac{355}{4}\,\mathrm{Log}[-2+x] - 105\,\mathrm{Log}[-1+x] - \frac{7\,\mathrm{Log}[x]}{4}$$

Another example

$$f = \frac{x^3\left(2\,x^2-1\right)}{x^4 - x^2 + 1}$$

Find poles of the denominator

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x /. Solve[x^4 - x^2 + 1 == 0, x]
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$$\left\{-(-1)^{1/6},\ (-1)^{1/6},\ -(-1)^{5/6},\ (-1)^{5/6}\right\}$$

Thus, the original function f can be rewritten as

$$f = \frac{x^3\left(2\,x^2-1\right)}{x^4 - x^2 + 1} = \left(x^3\left(2\,x^2-1\right)\right)\Big/\left(\left(x - \sqrt[6]{-1}\right)\left(x + \sqrt[6]{-1}\right)\left(x - (-1)^{5/6}\right)\left(x + (-1)^{5/6}\right)\right)$$

Next,

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Apart[(x^3 (2 x^2 - 1)) / ((x - (-1)^(1/6)) (x + (-1)^(1/6)) (x - (-1)^(5/6)) (x + (-1)^(5/6)))]
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$$\left(-1+2\,(-1)^{1/3}\right)\Big/\left(2\left(-1+(-1)^{2/3}\right)\left(1+(-1)^{2/3}\right)\left((-1)^{1/6}-x\right)\right)-\left((-1)^{1/3}\left(1+2\,(-1)^{2/3}\right)\right)\Big/\left(2\left(-1+(-1)^{2/3}\right)\left(1+(-1)^{2/3}\right)\left((-1)^{5/6}-x\right)\right)+2\,x+\left(1-2\,(-1)^{1/3}\right)\Big/\left(2\left(-1+(-1)^{2/3}\right)\left(1+(-1)^{2/3}\right)\left((-1)^{1/6}+x\right)\right)+\left((-1)^{1/3}\left(1+2\,(-1)^{2/3}\right)\right)\Big/\left(2\left(-1+(-1)^{2/3}\right)\left(1+(-1)^{2/3}\right)\left((-1)^{5/6}+x\right)\right)$$

Finally, integrating termby term, we get

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Integrate[#, x] & /@ %
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$$x^2+\left(\left(-2+(-1)^{1/3}\right)\left(2\,\mathrm{ArcTan}\left[\frac{\sqrt{3}\,x}{2\sqrt{3}-3\,x}\right]+i\,\mathrm{Log}\left[-1+\sqrt{3}\,x-x^2\right]\right)\right)\Big/\left(-6\,i+2\sqrt{3}\right)+\left(\sqrt{3}\left(2\,i\,\mathrm{ArcTan}\left[\frac{\sqrt{3}-3\,x}{-3+\sqrt{3}\,x}\right]+\mathrm{Log}\left[-1+\sqrt{3}\,x-x^2\right]\right)\right)\Big/\left(-6\,i+2\sqrt{3}\right)+\left(\left(2-(-1)^{1/3}\right)\left(2\,\mathrm{ArcTan}\left[\frac{\sqrt{3}\,x}{2\sqrt{3}+3\,x}\right]-i\,\mathrm{Log}\left[1+\sqrt{3}\,x+x^2\right]\right)\right)\Big/\left(-6\,i+2\sqrt{3}\right)+\left(\sqrt{3}\left(-2\,i\,\mathrm{ArcTan}\left[\frac{\sqrt{3}+3\,x}{3+\sqrt{3}\,x}\right]+\mathrm{Log}\left[1+\sqrt{3}\,x+x^2\right]\right)\right)\Big/\left(-6\,i+2\sqrt{3}\right)$$

Compare with

$$\mathtt{Integrate}\left[\frac{x^3\left(2\,x^2-1\right)}{x^4-x^2+1},\,x\right]$$

$$x^2-\frac{1}{2}\sqrt{3}\,\mathrm{ArcTan}\left[\frac{-1+2\,x^2}{\sqrt{3}}\right]+\frac{1}{4}\,\mathrm{Log}\left[1-x^2+x^4\right]$$

The computer algebra answer is much nicer!!

Differential Algebra

Definition. An *integral domain* is a commutative ring without zero divisors (note, 0 is a zero divisor in any ring)

A6: $a * b = 0 \Longrightarrow a = 0$ or $b = 0$

For example, in Z_6 we have $2 * 3 = 0$, so 2 and 3 are zero-divisors.

Examples of an integral domain

- the ring $\mathbb{Z}$
- the polynomial ring $\mathbb{Z}(x)$
- the ring $\mathbb{Z}_p$ if p is prime

Definition. A *field* is a commutative ring with identity in which every non-zero element has a multiplicative inverse.

Theorem. *Every field is an integral domain.*

Proof. In a field $a*b = 0$. Multiply it by a^{-1} to get $b = 0$.

Theorem. *Every finite integral domain is a field.*

Definition. A *characteristic* of ring (or field) is the number of times the identity element can be added to itself to get 0. In the case when this never can be produced, the ring or field is called to have a characteristic zero.

Examples,

Q, $\mathbb{R}$, $\mathbb{C}$, characteristic 0.

The field Z_p has characteristic p.

Theorem. *The characteristic of an integral domain is either 0 or a prime number.*

Proof. Let n be the smallest such that $n*1 = 0$, where $n = k*m$. It follows $(k*1)\,(m*1) = 0$. But in an integral domain either of them is zero, so $k*1 = 0$ or $m*1 = 0$. Contradiction, since n is the smallest.Thus, n is prime.

Definition. Let R be an integral domain and $D : R \to R$ such that

$$D(f+g) = D(f) + D(g)$$

$$D(f*g) = D(f)*g + f*D(g)$$

then D is called a *differential operator*. And the pair (R, D) is a *differentail algebra.*

If R is a field, we get a *differentail field*.

Definition. If f and $g \in R$ and $D(f) = g$, then we say that f is an *integral* of g and we write $f = \int g$.

The problem of indefnite integration is to compute the inverse D^{-1} of the differential opeartor.

Theorem. In a differential field (F, D)

1. $D(0) = D(1) = 0$
2. $D\left(\frac{f}{g}\right) = \frac{D(f)*g - f*D(g)}{g^2}$

3. $D(f^n) = n\, f^{n-1}\, D(f)$

Proof.

D(1) = D(1*1) = D(1)*1 + 1*D(1) = 2 D(1) $\longrightarrow$ D(1) = 0

D(1) = D(1+0) = D(1) + D(0) $\longrightarrow$ D(0) = 0

it follows 1).

To prove 2) consider

$$D\left(\frac{f}{g}\right) = D(f) * g^{-1} + f * D\left(g^{-1}\right)$$

We find $D\left(g^{-1}\right)$ from

$$0 = D(1) = D\left(g * g^{-1}\right) = D(g) * g^{-1} + g * D\left(g^{-1}\right)$$

thus

$$D\left(g^{-1}\right) = -D(g) * g^{-2}$$

Therefore,

$$D\left(\frac{f}{g}\right) = D(f) * g^{-1} + f * D\left(g^{-1}\right) = \frac{g * D(f) - f * D(g)}{g}$$

QED■

Is the differential field closed under the inverse operator?

Lemma. For the rational function $\frac{1}{x} \in Q(x)$ there does not exists a rational function $r \in Q(x)$ such that $D(r) = \frac{1}{x}$.

Proof. Suppose that $r = \frac{p}{q} \in Q(x)$, $D(r) = \frac{1}{x}$ where GCD$(p, q) = 1$. Then

$$\frac{1}{x} = D(r) = D\left(\frac{p}{q}\right) = \frac{D(p) * q - p * D(q)}{q^2}$$

It follows that

$$x * q * D(p) - x * p * D(q) = q^2$$

This means that x divides q^2 and therefore q.

Hence, we can write $q = x^n * w$, where $n \geq 1$ and GCD$(w, x) = 1$.

Substituting $q = x^n * w$ into the previous equation, we obtain

$$x^{n+1} * w * D(p) - x * p * D(x^n * w) = q^2 = x^{2n} * w^2$$

or

$$x^{n+1} * w * D(p) - n * x^n * p * w - x^{n+1} * p * D(w) = x^{2n} * w^2$$

Canceling by x^n

$$x * w * D(p) - n * p * w - x * p * D(w) = x^n * w^2$$

and then collecting terms

$$n * p * w = x * \left[w * D(p) - x * p * D(w) - x^{n-1} * w^2\right]$$

we deduce that x divides $p * w$ and because GCD$(w, x) = 1$, we must have that x divides p. Contradiction, x cannot divides both p and q. ■

This lemma motivates for a domain extension.

Let F be a field and G is a differential extension over F.

Definition. $u \in G$ is called *logarithmic* over F if there exists such $p \in F$ that $D(u) = D(p) / p$ and we write $u = \log(p)$.

Definition. $u \in G$ is called *algebraic* over F if there exists a polynomial $p \in F$ such that $p(u) = 0$.

For the differential field of rational functions, the indefniite integral can always be expressed in an extension field requiring only two types of extensions: logarithmic and algebraic number extensions.

Eamples.

. $$\int \frac{1}{(x+1)^2} \, dx = \frac{-1}{x+1} \in Q(x)$$

$$\int \frac{1}{x} \, dx = \log x \in Q(x, \log x)$$

$$\int \frac{1}{x^3+x} \, dx = \log x - \frac{1}{2} \log\left(x^2+1\right) \in Q\left(x, \log x, \log\left(x^2+1\right)\right)$$

$$\int \frac{1}{x^2-2} \, dx = \frac{\log\left(x-\sqrt{2}\right) - \log\left(x+\sqrt{2}\right)}{2\sqrt{2}} \in Q\left(\sqrt{2}\right)(x, \log x)$$

Definition. $u \in G$ is called *exponential* over F if there exists such $p \in F$ that $D(u) / u = D(p)$ and we write $u = \exp(p)$.

Definition. G is called *elementary extension* over F if it is logarithmic and/or algebraic and/or exponential.

Examples.

$$\int\left(\log^2(x) + \frac{1}{x\log(x)}\right) dx$$

$$2\,x - 2\,x\,\text{Log}[x] + x\,\text{Log}[x]^2 + \text{Log}[\text{Log}[x]]$$

$$\int \frac{1}{e^x + 1}\, dx$$

$$x - \text{Log}[1 + e^x]$$

$$\int \sqrt{x + \sqrt{x+1}}\, dx$$

$$\frac{1}{12}\sqrt{x + \sqrt{1+x}}\left(-3 + 8\,x + 2\sqrt{1+x}\right) + \frac{5}{8}\,\text{Log}\left[1 + 2\sqrt{1+x} + 2\sqrt{x + \sqrt{1+x}}\right]$$

This integral cannot be done in elementary functions:

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Integrate[Exp[-x^2], x]
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$$\frac{1}{2}\sqrt{\pi}\;\text{Erf}[x]$$

Squarefree Factorization

Definition. We say that f is *squarefree* if it has no proper quadratic divisors.

Definition.The *squarefree factorization* of $f(x)$ is

$$f(x) = \prod_{k=1}^{n} g_k(x)^k = g_1(x)\, g_2(x)^2\, g_3(x)^3 \ldots g_n(x)^n$$

where each g_i is a squarefree polynomial and $\text{GCD}(g_i,\ g_k) = 1$

The squarefree part of a polynomial can be calculated without actually factoring the polynomial into irreducibles. We will see how to do this for fields of characteristic zero.

Definition. A field F is of characteristic zero, if for all $a \in F$, $a \neq 0$ and $n \in \mathbb{Z}$, $n \neq 0$ we have $n\, a \neq 0$.

Lemma. Let F be a field of characteristic zero. Then f is square-free $\Longleftrightarrow$ $\text{GCD}(f,\ f') = 1$.

Example. Consider

$$f = x^6 + 2\,x^3 + 1$$

over Z_3.

$$D(f) = 6\,x^5 + 6\,x^2 = 0\ (\text{mod}\ 3)$$

■ Squarefree factorization algorithm

This is Musser's algorithm originall presented in

D. R. Musser, *Algorithms for Polynomial Factorization*, Ph.D. thesis, University of Wisconsin, 1971.

Take

$$f(x) = \prod_{k=1}^{n} g_k(x)^k$$

find derivative

$$f'(x) = \sum_{k=1}^{n} g_1(x) \ldots k\, g_k(x)^{k-1}\, g_k{}'(x) \ldots g_n(x)$$

Hence

$$c(x) = \text{GCD}(f(x),\ f'(x)) = \prod_{k=2}^{n} g_k(x)^{k-1}$$

Then

$$w(x) = \frac{f(x)}{\text{GCD}(f(x),\ f'(x))} = \prod_{k=1}^{n} g_k(x)$$

is a product of squarefree factors. Calculating (if $c(x)$ is not 1, because otherwise $f(x)$ is squarefree)

$$y(x) = \text{GCD}(c(x),\ w(x)) = \prod_{k=2}^{n} g_k(x)$$

and observing that

$$g_1(x) = \frac{w(x)}{y(x)}$$

or

$$g_1(x) = \frac{\frac{f(x)}{c(x)}}{\mathrm{GCD}\left(c(x),\ \frac{f(x)}{c(x)}\right)}$$

we find the first squarefree factor.

To find $g_2(x)$, we observe that it is the first factor of $c(x)$. Thus

$$f(x) \longleftarrow c(x)$$

$$\mathrm{new\_c}\,(x) = \mathrm{GCD}(c(x),\ c'(x)) = \prod_{k=3}^{n} g_k(x)^{k-2} = \frac{c(x)}{y(x)}$$

$$w(x) = \frac{c(x)}{\mathrm{GCD}(c(x),\ c'(x))} = \frac{c(x)}{\mathrm{new\_c}\,(x)} = \frac{c(x)}{\frac{c(x)}{y(x)}} = y(x)$$

In short

$$c(x) = \frac{c(x)}{y(x)}$$

$$w(x) = y(x)$$

$$y(x) = \mathrm{GCD}(c(x),\ w(x))$$

$$g_2(x) = \frac{w(x)}{y(x)}$$

Applying these recursively, we find all g_k

References

[1] Michael Berry, Why are special functions special, *Physics Today*, April 2001.

[2] K.O. Geddes, S.R. Czapor, and G. Labahn, *Algorithms for Computer Algebra*, Kluwer Academic Publishers, 1992.

[3] M. Bronstein, *Symbolic Integration - Transcendental Functions (Algorithms and Computations in Mathematics)*, Vol 1, Springer-Verlag, 1996.