Gröbner Bases

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Graph Coloring with Gröbner bases

Theorem. *The problem of computing a groebner basis is NP Complete.*

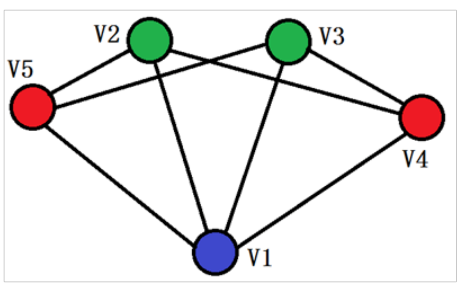
Show it's in NP. For a given finite set $G = \{g_1, g_2, \ldots, g_n\}$, we just need to use Buchberger's S-pair criterion: check if for any i and j, the remainder of division $S(g_i, g_j)$ by G is zero. This means, we need to perform the division algorithm $O(n^2)$ times. The division algorithm can be completed in polynomial time, so the whole check will finish in polynomial time.

Show it's NP hard. We should reduce a known NP complete problem to this problem in polynomial time. For simplicity, we choose graph 3-coloring problem.

For a given graph $G = \langle V, E \rangle$, we set $n = |V|$ variables named $x_1, \ldots, x_n$. We choose three colors as roots of the cubic equation $x_i^3 - 1$ for each vertex x_i. Two vertices on an edge must have different colors $x_i^3 = x_j^3$. This translates $x_i^3 - x_j^3 = (x_i - x_j)(x_i^2 + x_i x_j + x_j^2) = 0$. Therefore, we assign $x_i^2 + x_i x_j + x_j^2$ for each edge in E.

Next we compute the reduece Gröbner basis. If the basis is <1> - there is no coloring. Otherwise, we choose color accordingly.

Example.



In[8]:=
```
A = {{0, 1, 1, 1, 1}, {1, 0, 0, 1, 1}, {1, 0, 0, 1, 1}, {1, 1, 1, 0, 0}}
```

In[27]:=
```
basis = {}; n = 5;
For[i = 1, i ≤ n, i++, AppendTo[basis, x[i]^3 - 1]]
```

In[29]:=
```
For[i = 1, i < n, i++,
 For[j = i + 1, j ≤ n, j++,
  If[A[[i, j]] == 1, AppendTo[basis, x[i]^2 + x[i] x[j] + x[j]^2]]]]
```

In[77]:=
```
basis
```

Out[77]=
```
{-1 + x[1]^3, -1 + x[2]^3, -1 + x[3]^3, -1 + x[4]^3,
 -1 + x[5]^3, x[1]^2 + x[1] x[2] + x[2]^2, x[1]^2 + x[1] x[3] + x[3]^2,
 x[1]^2 + x[1] x[4] + x[4]^2, x[1]^2 + x[1] x[5] + x[5]^2, x[2]^2 + x[2] x[4] + x[4]^2,
 x[2]^2 + x[2] x[5] + x[5]^2, x[3]^2 + x[3] x[4] + x[4]^2, x[3]^2 + x[3] x[5] + x[5]^2}
```

In[78]:=
```
gb = GroebnerBasis[basis, Array[x, n]]
```

Out[78]=
```
{-1 + x[5]^3, x[4] - x[5],
 x[3]^2 + x[3] x[5] + x[5]^2, x[2] - x[3], x[1] + x[3] + x[5]}
```

The reduce Gröbner basis: $\{x_5{}^3 - 1,\ x_4 - x_5,\ x_3{}^2 + x_3\, x_5 + x_5{}^2,\ x_2 - x_3,\ x_1 + x_3 + x_5\}$

■ Graph Coloring Ideal

Define

$$V_{n,k} = \ < x_i^k - 1,\ x_i\ \in V >$$

$$E_{n,k} = \ < x_i^{k-1} + x_i^{k-2}\, x_j + \ldots + \ x_i\, x_j^{k-2} + x_j^{k-1},\ (x_i,\, x_j)\ \in E >$$

Then $V_{n,k} + E_{n,k}$ is called the k-coloring ideal.

An advantage of Gröbner bases is that it allows us to decide if a graph G is uniquely k-colorable up to the permutation.

Theorem. *A graph G with n* vertices $x_1,\ x_2,\ x_3,\ z_1,\ z_2,\ \ldots,\ z_{n-3}$ *is uniquely 3-colorable iff a correspondent Gröbner basis has the following form*

$$< x_3{}^3 - 1,\, x_2{}^2 + x_2\, x_3 + x_3{}^2,\ x_1 + x_2 + x_3,\ f(z_1),\ \ \ldots f(z_{n-3}) >$$

where

$$f(z_i) = \begin{cases} z_i - x_j & j = 2,\ 3,\ \text{and } z_i \text{ has the same color as } x_j \\ z_i + x_2 + x_3 & \text{if } z_i \text{ has the same color as } x_1 \end{cases}$$

The uniqueness follows from the fact that the first three equations in the Groebner basis uniquely define colors for three vertices, the other linear equations $f(z_i)$ uniquely define the rest.

■ Demo

In[21]:=
```
ChooseColor[x_] :=
```

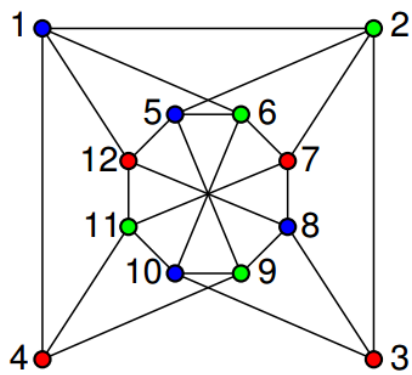
```
  Module[{}, Switch[x, 1, Red, -(-1)^(1/3), Green, -1 + (-1)^(1/3), Blue,
    _, Black]];

GraphColoring[g_Graph] := Module[
  {i, j, n, rg, A, basis = {}, gb, tmp, ansRule = {}, ansList = {}, cg},
  (* rg: random generated graph; cg: color graph *)

  rg = g;
  n = VertexCount[g];

  cg = Graph[Range[n], EdgeList[rg], VertexLabels → Table[i → V_i, {i, n}],
    VertexStyle → Black];

  (* Get the adjacency matrix *)
  A = Normal[AdjacencyMatrix[rg]];

  (* Construct the 3-coloring ideal *)
  For[i = 1, i ≤ n, i++, AppendTo[basis, x[i]^3 - 1]];
  For[i = 1, i < n, i++,
   For[j = i + 1, j ≤ n, j++,
    If[A[[i, j]] == 1, AppendTo[basis, x[i]^2 + x[i] x[j] + x[j]^2]];
   ]
  ]; (* I guess there is a more elegant way... *)

  gb = GroebnerBasis[basis, Array[x, n]];
  (* calculate the Groebner basis of 3-coloring ideal wrt lex order *)
  If[TrueQ[gb[[1]] == 1], Print["No solution"];
   Return[{rg, A, gb, ansRule, ansList, cg}]];
  (* The graph is not 3-colorable iff 3-coloring ideal = <1> *)

  tmp = gb;
  For[i = 1;
   j = n, j ≥ 1 && i ≤ Length[gb], i++,
   (* Use i to pick polynomials.  Use j to pick variables. *)
   If[TrueQ[tmp[[i]] == 0], (* If tmp[[i]] is 0, do nothing;
    else solve x[j] and back substitute x[j] to GB.  Always
     choose the first root. *)
    ,
    rule = First[Solve[tmp[[i]] == 0, x[j]]];
    tmp = Simplify[tmp /. rule];
    PrependTo[ansRule, rule];
    PrependTo[ansList, (rule /. Rule → List)[[1, 2]] // FullSimplify]; j-- ]
```

```
  ];
  cg = Graph[Range[n], EdgeList[rg], VertexLabels → Table[i → Vi, {i, n}],
    VertexStyle → Table[i → ChooseColor[ansList[[i]]], {i, n}]];
  {rg, A, gb, ansRule, ansList, cg}
 ]
```

In[23]:=
```
Graph1 = Graph[Range[9], {1 ⟷ 2, 1 ⟷ 4, 1 ⟷ 5, 1 ⟷ 6, 2 ⟷ 7, 2 ⟷ 3,
    2 ⟷ 5, 3 ⟷ 4, 3 ⟷ 6, 3 ⟷ 9, 5 ⟷ 6, 6 ⟷ 8, 7 ⟷ 8, 8 ⟷ 9},
   VertexLabels → Table[i → Vi, {i, 9}]];
```

In[24]:=
```
GraphColoring[Graph1]
```

In 2001 Chao provided a 12-vertex 22-edges graph that uniquely colored.



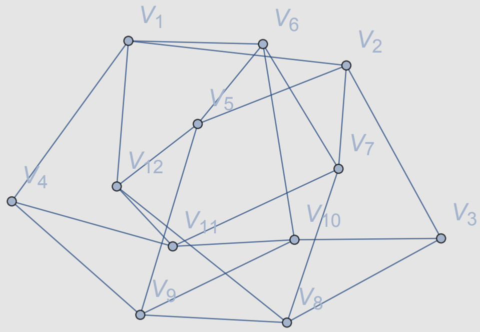
In[25]:=
```
ChaoChenGraph =
  Graph[Range[12], {1 ⟷ 2, 1 ⟷ 6, 1 ⟷ 12, 1 ⟷ 4, 2 ⟷ 5, 2 ⟷ 7, 2 ⟷ 3,
    3 ⟷ 8, 3 ⟷ 10, 4 ⟷ 9, 4 ⟷ 11, 5 ⟷ 6, 6 ⟷ 7, 7 ⟷ 8, 8 ⟷ 9, 9 ⟷ 10,
    10 ⟷ 11, 11 ⟷ 12, 12 ⟷ 5, 5 ⟷ 9, 6 ⟷ 10, 7 ⟷ 11, 8 ⟷ 12},
   VertexLabels → Table[i → Vi, {i, 12}]];
```

In[79]:=

```
GraphColoring[ChaoChenGraph]
```

Out[79]=

{



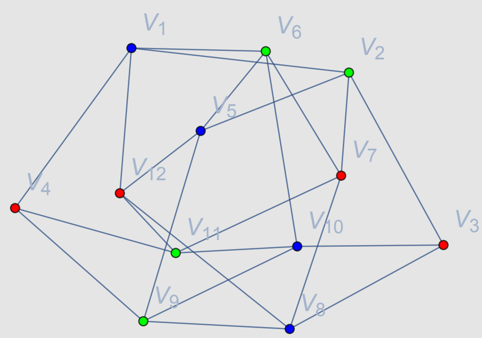
,

```
{{0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1}, {1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0},
 {0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0},
 {0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1}, {1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0},
 {0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0}, {0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1},
 {0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0}, {0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0},
 {0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1}, {1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0}},
```

$\{-1 + x[12]^3,\ x[11]^2 + x[11]\,x[12] + x[12]^2,\ x[10] + x[11] + x[12],\ x[9] - x[11],$
$x[8] + x[11] + x[12],\ x[7] - x[12],\ x[6] - x[11],\ x[5] + x[11] + x[12],$
$x[4] - x[12],\ x[3] - x[12],\ x[2] - x[11],\ x[1] + x[11] + x[12]\},$

$\{\{x[1] \to -1 + (-1)^{1/3}\},\ \{x[2] \to -(-1)^{1/3}\},\ \{x[3] \to 1\},\ \{x[4] \to 1\},$
$\{x[5] \to -1 + (-1)^{1/3}\},\ \{x[6] \to -(-1)^{1/3}\},\ \{x[7] \to 1\},\ \{x[8] \to -1 + (-1)^{1/3}\},$
$\{x[9] \to -(-1)^{1/3}\},\ \{x[10] \to -1 + (-1)^{1/3}\},\ \{x[11] \to -(-1)^{1/3}\},\ \{x[12] \to 1\}\},$

$\{-1 + (-1)^{1/3},\ -(-1)^{1/3},\ 1,\ 1,\ -1 + (-1)^{1/3},\ -(-1)^{1/3},\ 1,\ -1 + (-1)^{1/3},$
$-(-1)^{1/3},\ -1 + (-1)^{1/3},\ -(-1)^{1/3},\ 1\},$



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References

[1] C. J. Hillar and D, Windfeldt, *Algebraic Characterization of Uniquely vertex Colorable Graphs*, J. Comb. Theory., 98(2), 2008, 400-414.