Petkovšek's algorithm

"All the wonders of our universe can in effect be captured by simple rules, yet [...] there can be no way to know all the consequences of these rules, except in effect just to watch and see how they unfold."


Algorithm

The algorithm deals with finding a hypergeometric term solution to

$$\sum_{j=0}^{N} d_j(n) S(n + j) = 0$$

where $d_j$ are polynomials in $n$. For simplicity, let us consider a second order equation

$$d_2(n) S(n + 2) + d_1(n) S(n + 1) + d_0(n) S(n) = 0 \quad (1)$$

where $S(n)$ is assumed to be a hypergeometric term

$$\frac{S(n + 1)}{S(n)} \in Q(n)$$

If all $d_k(n)$ are constants then we solve the equation by means of the characteristic equation

$$d_2 \lambda^2 + d_1 \lambda + d_0 = 0$$

The roots of this equation define the general solution

$$S(n) = c_1 \lambda_1^n + c_2 \lambda_2^n$$

If $d_k(n)$ is polynomials in $n$ then proceed in the following way. Divide (1) by $S(n)$
\[ d_2(n) \frac{S(n + 2)}{S(n)} + d_1(n) \frac{S(n + 1)}{S(n)} + d_0(n) = 0 \]

or

\[ d_2(n) \frac{S(n + 2)}{S(n + 1)} \frac{S(n + 1)}{S(n)} + d_1(n) \frac{S(n + 1)}{S(n)} + d_0(n) = 0 \]  \hspace{1cm} (2)

From the Gosper’s algorithm, we know that any rational function can be represented as

\[ \frac{S(n + 1)}{S(n)} = c \frac{p(n + 1)}{p(n)} \frac{q(n + 1)}{r(n + 1)} \]  \hspace{1cm} (3)

where \( p(n), \ q(n), \ r(n) \) are monic (the leading coefficient is 1) and coprime

\[ \text{GCD}(q(n), \ r(n + j)) = 1, \ j \in \mathbb{N}_0. \]

\[ \text{GCD}(p(n), \ q(n + 1)) = 1, \]

\[ \text{GCD}(p(n), \ r(n)) = 1, \]

Substituting (3) into (2), yields

\[ d_2(n) c \frac{p(n + 2)}{p(n + 1)} \frac{q(n + 2)}{r(n + 2)} c \frac{p(n + 1)}{p(n)} \frac{q(n + 1)}{r(n + 1)} + d_1(n) c \frac{p(n + 1)}{p(n)} \frac{q(n + 1)}{r(n + 1)} + d_0(n) = 0 \]

Multiply it by \( p(n) r(n + 1) r(n + 2) \) to get

\[ c^2 d_2(n) \ p(n + 2) \ q(n + 1) \ q(n + 2) + \]

\[ c \ d_1(n) \ p(n + 1) \ q(n + 1) \ r(n + 2) + \]

\[ d_0(n) \ p(n) \ r(n + 1) \ r(n + 2) = 0 \]  \hspace{1cm} (4)

Remember, our goal is to find \( p(n), \ q(n), \ r(n) \) and constant \( c \).

The logic in the next paragraph is somewhat similar to the proof of step 2 in Gosper’s algorithm.

The first two terms of (4) are divisible by \( q(n + 1) \), therefore the last term \( d_0(n) \ p(n) \ r(n + 1) \ r(n + 2) \) must be divisible by \( q(n + 1) \). Since \( p, \ q \) and \( r \) are relatively prime, then \( q(n + 1) \) must divide \( d_0(n) \). In other words, \( q(n + 1) \) must be a factor of \( d_0(n) \). This leads us to the finite number of choices for \( q(n + 1) \).

The last two terms of (4) are divisible by \( r(n + 2) \), therefore the first term must be divisible by \( r(n + 2) \). Since \( p, \ q \) and \( r \) are relatively prime, then \( r(n + 2) \) must divide \( d_2(n) \). In other words, \( r(n + 2) \) must be a factor of \( d_2(n) \). This leads us to finite number of choices for \( r(n + 2) \).

Once we know \( q \) and \( r \), we can easily find a rational constant \( c \). Divide (4) by \( q(n + 1) \ r(n + 2) \)
This gives us an equation with polynomial coefficients. Equating the leading coefficient to zero, generates a quadratic equation for \( c \). But \( p(n) \) is still unknown. ??? hmm... Where is a catch? \( p(n) \) is a monic polynomial.

Now the last step.

So far we found the constant \( c \) and two polynomials \( q \) and \( r \). To find the polynomial \( p \) we must solve (5). Since all coefficients in (5) are polynomials, we need to find a polynomial solution.

Let us consider a generic equation of the second order with polynomial coefficients

\[
a_2(n) \, Y(n + 2) + a_1(n) \, Y(n + 1) + a_0(n) \, Y(n) = 0
\]

We need to find an upper bound for the degree of a polynomial solution. Assume the following

\[
a_2(n) = \alpha_p \, n^p + \alpha_{p-1} \, n^{p-1} + ... \\
a_1(n) = \beta_p \, n^p + \beta_{p-1} \, n^{p-1} + ... \\
a_0(n) = \gamma_p \, n^p + \gamma_{p-1} \, n^{p-1} + ...
\]

where \( P \) is the maximal degree of \( a_2, a_1 \) and \( a_0 \), and all coefficients \( \alpha_j, \beta_j, \gamma_j \) are known. We are looking for a monic polynomial solution

\[
Y(n) = n^M + \delta_{M-1} n^{M-1} + ... \\
Y(n + 1) = n^M + (M + \delta_{M-1}) n^{M-1} + ... \\
Y(n + 2) = n^M + (2M + \delta_{M-1}) n^{M-1} + ... 
\]

where order \( M \) and coefficients \( \delta_k \) are to be determined. Substitute these into the difference equation and take coefficients of the first three dominant terms. We obtain

\[
n^M + P : \alpha_p + \beta_p + \gamma_p \\
n^M + P - 1 : \alpha_{p-1} + \beta_{p-1} + \gamma_{p-1} + M \, (2 \, \alpha_p + \beta_p) + (2 \, \alpha_p + \beta_p + \gamma_p) \, \delta_{M-1} \\
n^M + P - 2 : \frac{1}{2} \left( 4 \, \alpha_p + \beta_p \right) M^2 + \left( 2 \, \alpha_{p-1} + \beta_{p-1} - 2 \, \alpha_p - \frac{\beta_p}{2} + (2 \, \alpha_p + \beta_p) \, \delta_{M-1} \right) M + \alpha_{p-2} + \beta_{p-2} + \gamma_{p-2} + (2 \, \alpha_p + \beta_p + \gamma_p) \, \delta_{M-2} + (\alpha_{p-1} - 2 \, \alpha_p + \beta_p - \gamma_p) \cdot \delta_{M-1}
\]

Each of them must be zero. Start with the first

\[
\alpha_p + \beta_p + \gamma_p = 0 \tag{6}
\]

If this condition is not satisfied, then no polynomial solution exists. Suppose (6) is satisfied, then the
next coefficient gives
\[ \alpha_{p-1} + \beta_{p-1} + \gamma_{p-1} + M (2 \alpha_p + \beta_p) + (\alpha_p + \beta_p + \gamma_p) \delta_{M-1} = 0 \]
or
\[ \alpha_{p-1} + \beta_{p-1} + \gamma_{p-1} + M (2 \alpha_p + \beta_p) = 0 \]  
(7)
This splits into two subcases
1) case
\[ 2 \alpha_p + \beta_p \neq 0 \]
then the order \( M \) is uniquely defined from (7)
\[ M = \frac{-\alpha_{p-1} + \beta_{p-1} + \gamma_{p-1}}{2 \alpha_p + \beta_p} \]  
(8)
2) case
\[ 2 \alpha_p + \beta_p = 0 \]
Then
\[ \alpha_{p-1} + \beta_{p-1} + \gamma_{p-1} = 0 \]  
(9)
and we must look at the next coefficient (a coefficient by \( n^{M+P-2} \)) that is
\[ \alpha_p M^2 + (2 \alpha_{p-1} + \beta_{p-1} - \alpha_p) M + \alpha_{p-2} + \beta_{p-2} + \gamma_{p-2} = 0 \]  
(10)
Observe, that \( \alpha_p \neq 0 \), because otherwise \( \beta_p = 0 \) and then by (6) \( \gamma_p = 0 \), which will contradict to the assumption that \( p \) has a maximal degree of \( p_2, \ p_1 \) and \( p_0 \). Therefore, equation (10) has two solutions, one (or both) defines the upper degree of a polynomial solution.

**Example**

Find a hypergeometric term solution to
\[ 9 (n+2) S(n+2) - 3 (n+4) S(n+1) - 2 (n+3) S(n) = 0 \]
\[ S(0) = S(1) = 1 \]
We assume that
\[ \frac{S(n+1)}{S(n)} = c \frac{p(n+1)}{p(n)} \frac{q(n+1)}{r(n+1)} \]
where
\[ q(n+1) \text{ must divide } d_0(n) \]
\[ r(n+2) \text{ must divide } d_2(n) \]
Here are our choices

\[ q(n + 1) \text{ is either } 1 \text{ or } n + 3 \]
\[ r(n + 2) \text{ is either } 1 \text{ or } n + 2 \]

\( \blacksquare \) **q(n+1) = 1; r(n+2)=1**

We find constant \( c \) from the equation (since \( p, q \) and \( r \) are monic, we replace them by 1 in equation (5))

\[ c^2 d_2(n) + c d_1(n) + d_0(n) = 0 \]

that simplifies to

\[ 9(n + 2) c^2 - 3(n + 4) c - 2(n + 3) = 0 \]
\[ n(9 c^2 - 3 c - 2) + 6(3 c^2 - 2 c - 1) = 0 \]

Solving

\[ 9 c^2 - 3 c - 2 = 0 \]

we obtain

\[ c = -\frac{1}{3} \text{ or } c = \frac{2}{3} \]

Note, constant \( c \) is defined only by a leading coefficient of equation (5).

**Case 1.** \( c = -\frac{1}{3} \)

We need to find a polynomial solution to

\[ (n + 2) p(n + 2) + (n + 4) p(n + 1) - 2(n + 3) p(n) = 0 \]

Compute

\[ \alpha_p + \beta_p + \gamma_p = 1 + 1 - 2 = 0 \]
\[ \alpha_{p-1} + \beta_{p-1} + \gamma_{p-1} + M(2 \alpha_p + \beta_p) = 2 + 4 - 6 + M(2 + 1) = 0 \]

The solution \( p(n) \) is a constant, since \( M = 0 \). Therefore, we have

\[ \frac{S(n + 1)}{S(n)} = c \frac{p(n + 1)}{p(n)} \frac{q(n + 1)}{r(n + 1)} = -\frac{1}{3} \frac{p(n + 1)}{p(n)} \frac{1}{1} = -\frac{1}{3} \]

or

\[ Y(n) = \left( -\frac{1}{3} \right)^n \]

**Case 2.** \( c = \frac{2}{3} \)

We find an upper bound on the polynomial solution
The first condition (the sum of leading coefficients should be zero)

\[ \alpha_p + \beta_p + \gamma_p = 0 \]

The next condition

\[ \alpha_{p-1} + \beta_{p-1} + \gamma_{p-1} + M (2 \alpha_p + \beta_p) = 0 \]

which is

\[ 3M - 3 = 0 \]

So, the solution is linear

\[ p(n) = n + x \]

where \( x \) is unknown. We find \( x \) by substituting \( p(n) \) into the original equation

\[ (2n + 4)(n + 2 + x) - (n + 4)(n + 1 + x) - (n + 3)(n + x) = 0 \]

This simplifies to

\[ 4 - 3x = 0 \]

Thus,

\[ p(n) = n + \frac{4}{3} \]

And

\[
\frac{S(n + 1)}{S(n)} = c \frac{p(n + 1)}{p(n)} \frac{q(n + 1)}{r(n + 1)} = 2 \frac{n + \frac{7}{3}}{n + \frac{4}{3}} \frac{1}{1} = \frac{2}{3} \frac{3n + 7}{3n + 4}
\]

or

\[ S(n) = \frac{3n + 4}{4} \left( \frac{2}{3} \right)^n \]

Finally, we combine the above two cases to get a general solution

\[ S(n) = c_1 \left( -\frac{1}{3} \right)^n + c_2 \frac{3n + 4}{4} \left( \frac{2}{3} \right)^n \]

where unknown \( c_1 \) and \( c_2 \) can be easily found from the initial conditions

\[ S(0) = S(1) = 1 \]

We get a system of two linear equations

\[ c_1 + c_2 = 1 \]

\[ c_1 \left( -\frac{1}{3} \right) + c_2 \frac{7}{4} \left( \frac{2}{3} \right) = 1 \]
solving each, yields

\[
c_1 = \frac{1}{9} \quad \text{and} \quad c_2 = \frac{8}{9}
\]

Hence,

\[
S(n) = \frac{1}{9} \left( -\frac{1}{3} \right)^n + \frac{8}{9} \frac{3n + 4}{4} \left( \frac{2}{3} \right)^n
\]

**q(n+1) = 1; r(n+2)=n+2**

Equation for constant \( c \)

\[-2n^2 + (-3c - 8)n + 3(3c^2 - 4c - 2) = 0\]

Such constant \( c \) does not exist

**q(n+1) = n+3; r(n+2)=1**

Equation for constant \( c \)

\[9c^2n^2 + 3c(18c - 1)n + 2(36c^2 - 6c - 1) = 0\]

It has only a trivial solution \( c = 0 \).

**q(n+1) = n+3; r(n+2)=n+2**

This case cannot be chosen, since they have a common polynomial GCD:

```plaintext
q[n_] := n + 2;
r[n_] := n
Table[PolynomialGCD[q[n], r[n + j]], {j, 0, 5}]

{1, 1, 2 + n, 1, 1, 1}
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**References**
