Computer Science 355

Modern Computer Algebra

Assignment 4

solutions

Problem 1

Let $X = C([0,\ 1])$ be the set of all continuous functions with domain $[0,\ 1]$, and range $\mathbb{R}$. Further impose a ring structure on this set by making $*$ and $+$ function multiplication and addition. Prove that the set $M = \{f \in X \mid f(1) = 0\}$ is an ideal in this ring.

First show that M is an ideal using the ideal test. $M \neq 0$, because the constant 0 function, $f(x) = 0,\ \forall\, x \in M$. Let $f,\ g \in M$. Then $(\mathrm{f0g})\,(1) = f(1) - g(1) = 0$, so $f - g \in M$. Now let $f \in M$ and $r \in X$. We have $(r\,f)\,(1) = (f\,r)\,(1) = f\,(1)\,r\,(1) = 0$. Thus M is an ideal.

Problem 2

Given two ideals I and J of a ring R with 1 such that $I + J = R$. Show that there is some $x \in R$ such that $x - a \in I$ and $x - b \in J$ for all $a,\ b\ \in R$.

As $I + J = R$, we have $r \in I$ and $s \in J$ such that $r + s = 1$.

Then $s - 1 = -r \in I$, and $r - 1 = -s \in J$.

Let $x = a\,s + b\,r \in R$, then $x - a = a\,(s - 1) + b\,r \in I$

and $x - b = a\,s + b\,(r - 1) \in J$.

Problem 3

Determine whether a given polynomial is in an ideal

1. $x^3 + 2\,x^2 + 2\,x + 1,\quad I = \{x + 1\}$
2. $x^2 - 4\,x + 4,\quad I = \{x^3 - 6\,x^2 + 12\,x - 8,\ 2\,x^3 - 10\,x^2 + 16\,x - 8\}$
3. $x^3 - 1,\quad I = \{x^9 - 1,\ x^5 + x^3 - x^2 - 1\}$

1. Polynomial $x^3 + 2\,x^2 + 2\,x + 1$ is in ideal.$\{x + 1\}$, since

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Factor[x^3 + 2 x^2 + 2 x + 1]
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$(1 + x)\ (1 + x + x^2)$

2. Polynomial $x^2 - 4x + 4$ is in ideal $\{x^3 - 6x^2 + 12x - 8,\ 2x^3 - 10x^2 + 16x - 8\}$, since

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Expand[-(x^3 - 6 x^2 + 12 x - 8) + 1/2 (2 x^3 - 10 x^2 + 16 x - 8)]
```

$4 - 4x + x^2$

Another proof

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PolynomialGCD[x^3 - 6 x^2 + 12 x - 8, 2 x^3 - 10 x^2 + 16 x - 8]
```

$4 - 4x + x^2$

3. Polynomial $x^3 - 1$ is in ideal $\{x^9 - 1,\ x^5 + x^3 - x^2 - 1\}$

First proof

$$x^9 - 1 = (x^3 - 1)(x^6 + x^3 + 1)$$

$$x^5 + x^3 - x^2 - 1 = (x^3 - 1)(x^2 + 1)$$

Second proof

```
Expand[(x^9 - 1) x +
  (x^5 + x^3 - x^2 - 1) (-x^5 + x^3 - x^2 - x + 1)]
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$-1 + x^3$

Problem 4

Working in $Q[x,\ y,\ z]$ and using the lex order with $x > y > z$, prove that $\{x - y,\ y^2 + z\}$ is a Gröbner basis

Compute S-polynomial

$$S(f,\ g) = \frac{x\,y^2}{x}\,f - \frac{x\,y^2}{y^2}\,g = y^2\,f - x\,g = -x\,z - y^3$$

and then reduce it wrt the basis

$$-x\,z - y^3\ \to_{x-y} = -y^3 - y\,z \to_{y^2+z}\ = 0$$

Problem 5

Find a Gröbner basis for $< z + y\,x^2,\; z\,x + y >$ with respect to lex with $x < y < z$.

Compute S-polynomial

$$S(f_1,\, f_2) = \frac{z\,x}{z}\, f_1 - \frac{z\,x}{z\,x}\, f_2 = y\,x^3 - y$$

Adjust the basis:

$$< z + y\,x^2,\; z\,x + y,\; y\,x^3 - y >$$

Compute S-polynomial

$$S(f_2,\, f_3) = \frac{z\,y\,x^3}{z}\, f_2 - \frac{z\,y\,x^3}{z\,x}\, f_3 = z\,y\; + \; y^2\,x^2$$

$$S(f_2,\, f_3) \rightarrow_{f_1} = 0$$

Skip $S(f_1,\; f_3)$, since

$$\text{LCM}(\text{LM}(f_1),\, \text{LM}(f_3)) \;=\; \text{LM}(f_1),\, \text{LM}(f_3))$$

Reduce it wrt the basis, to get

$$< z + y\,x^2,\; y\,x^3 - y >$$

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GroebnerBasis[{x^2 y + z, x z + y}, {z, y, x}]
{-y + x^3 y, x^2 y + z}
```

Problem 6

Find a Gröbner basis for $< x^2\,y + z,\; x\,z + y >$ with respect to lex with $x > y > z$.

Compute S-polynomial

$$S(f_1,\, f_2) = \frac{x^2\,y\,z}{x^2\,y}\, f_1 - \frac{x^2\,y\,z}{x\,z}\, f_2 = x\,y^2 - z^2$$

Adjust the basis:

$$< x^2\,y + z,\; x\,z + y,\; x\,y^2 - z^2 >$$

Compute S-polynomial

$$S(f_2,\, f_3) = \frac{x\,y^2\,z}{x\,z}\, f_2 - \frac{x\,y^2\,z}{x\,y^2}\, f_3 = y^3 + z^3$$

Adjust the basis:

$$< x^2\, y + z,\ \ x\, z + y,\ x\, y^2 - z^2,\ y^3 + z^3 >$$

Compute S-polynomial

$$S(f_3,\ f_4) = \frac{x\, y^3}{x\, y^2}\, f_3 - \frac{x\, y^3}{y^3}\, f_4 = -x\, z^3 - y\, z^2$$

$$S(f_3,\ f_4) \to_{f_2} = 0$$

Compute S-polynomial

$$S(f_1,\ f_3) = \frac{x^2\, y^2}{x^2\, y}\, f_1 - \frac{x^2\, y^2}{x\, y^2}\, f_3 = x\, z^2 + y\, z$$

$$S(f_1,\ f_3) \to_{f_2} = 0$$

Compute S-polynomial

$$S(f_1,\ f_4) = \frac{x^2\, y^3}{x^2\, y}\, f_1 - \frac{x^2\, y^3}{y^3}\, f_4 = -x^2\, z^3 + y^2\, z$$

$$S(f_1,\ f_4) \to_{f_2} = 0$$

The Groebner basis

$$< x^2\, y + z,\ \ x\, z + y,\ x\, y^2 - z^2,\ y^3 + z^3 >$$

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GroebnerBasis[{x^2 y + z, x z + y}, {x, y, z}]
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{y^3 + z^3, y + x z, x y^2 - z^2, x^2 y + z}
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