Computer Science 355
Modern Computer Algebra

Assignment 1
solutions

Problem 1 (Programming) (15 pts)

Let $d(n)$ be defined as the sum of proper divisors of n (numbers less than n which divide evenly into n). If $d(a) = b$ and $d(b) = a$, where $a \neq b$, then a and b are an amicable pair and each of a and b are called amicable numbers.

For example, the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110; therefore $d(220) = 284$. The proper divisors of 284 are 1, 2, 4, 71 and 142; so $d(284) = 220$.

Your task is to find all the amicable numbers for given n and return their sum. *Mathematica* functiuons: Divisors, Most, Sum.

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```
AmicableSum[n_Integer?Positive] :=
 Module[{sum = 0, k, x, y},
  For[k = 2, k < n, k++,
   x = Total@Most@Divisors[k];
   y = Total@Most@Divisors[x];
   If[y == k, sum += k]
  ];
  sum
 ]
```

```
AmicableSum[10 000]
```

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Problem 2 (Difference equations)

Solve the recurrence

$$a_n - 5\,a_{n-1} + 7\,a_{n-2} - 3\,a_{n-3} = 0$$
$$a_0 = 1,\; a_1 = 2,\; a_2 = 3$$

using the characteristic equation. Verify your solution with *Mathematica's* RSolve[].

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The charactereistic equation

$$\lambda^3 - 5\,\lambda^2 + 7\,\lambda - 3 = 0$$

has three real roots

```
Solve[λ^3 - 5 λ^2 + 7 λ - 3 == 0, λ]
```

```
{{λ → 1}, {λ → 1}, {λ → 3}}
```

Thus, the general solution is given by

$$a_n = c_1\,3^n + c_2 + c_3\,n$$

We find c_k from the initial condiitons

$$a_0 = c_1 + c_2 = 1$$

$$a_1 = c_1\,3 + c_2 + c_3 == 2$$

$$a_2 = c_1\,9 + c_2 + c_3\,2 == 3$$

with *Mathematica*

```
Solve[{c1 + c2 == 1, 3 c1 + c2 + c3 == 2, 9 c1 + c2 + 2 c3 == 3},
  {c1, c2, c3}]
```

```
{{c1 → 0, c2 → 1, c3 → 1}}
```

Therefore, the solution is

$$a_n = 1 + n$$

Verify,

```
RSolve[{a[n] - 5 a[n - 1] + 7 a[n - 2] - 3 a[n - 3] == 0,
  a[0] == 1, a[1] == 2, a[2] == 3}, a[n], n]
```

```
{{a[n] → 1 + n}}
```

Problem 3 (Gamma Function)

Using the functional properties of the Gamma functions (see the Lecture 2), compute

$$\frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{11}{3}\right)}, \quad \frac{\Gamma\left(-\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}, \quad \Gamma\left(-\frac{3}{2}\right)$$

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We shall use the following property

$$\Gamma(x+1) = x\,\Gamma(x)$$

For the first one we have

$$\frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{11}{3}\right)} = \frac{\Gamma\left(\frac{5}{3}\right)}{\frac{8}{3}\,\Gamma\left(\frac{8}{3}\right)} = \frac{\Gamma\left(\frac{5}{3}\right)}{\frac{8}{3}\,\frac{5}{3}\,\Gamma\left(\frac{5}{3}\right)} = \frac{9}{40}$$

For the second

$$\frac{\Gamma\left(-\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} = \frac{\Gamma\left(-\frac{3}{4}\right)}{\frac{1}{4}\,\Gamma\left(\frac{1}{4}\right)} = \frac{\Gamma\left(-\frac{3}{4}\right)}{\frac{1}{4}\left(\frac{-3}{4}\right)\Gamma\left(\frac{-3}{4}\right)} = -\frac{16}{3}$$

For the last we have add another functional property

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin(\pi\,x)}, \quad 0 < x < 1$$

using which we can find $\Gamma\left(\frac{1}{2}\right)$

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin\left(\pi\,\frac{1}{2}\right)}$$

$$\Gamma\left(\frac{1}{2}\right)^2 = \pi$$

Therefore,

$$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = \frac{4\sqrt{\pi}}{3}$$

Problem 4 (Celine's algorithm)

Evaluate the sum using Sister Celine' s algorithm

$$\sum_{k=0}^{n} \binom{k+n}{k} 2^{-k}$$

Demonstrate each step of the algorithm.Find a recurrence for the summand,
then sum that recurence over the range to find a recurrence
that is satisfied by the sum.Finally, solve that recurrence.

```
s[n_, k_] := Binomial[n + k, k] 2^(-k)
```

$$a + b\,\frac{s[n+1,k]}{s[n,k]} + c\,\frac{s[n,k+1]}{s[n,k]} + d\,\frac{s[n+1,k+1]}{s[n,k]}$$

$$a + \frac{b\,\text{Binomial}[1+k+n,k]}{\text{Binomial}[k+n,k]} + \frac{c\,\text{Binomial}[1+k+n,1+k]}{2\,\text{Binomial}[k+n,k]} + \frac{d\,\text{Binomial}[2+k+n,1+k]}{2\,\text{Binomial}[k+n,k]}$$

```
FunctionExpand[%]
```

$$a + \frac{c\,(1+k+n)}{2\,(1+k)} + \frac{b\,(1+k+n)}{1+n} + \frac{d\,(1+k+n)\,(2+k+n)}{2\,(1+k)\,(1+n)}$$

```
Numerator[Together[%]]
```

$$2a + 2b + c + 2d + 2ak + 4bk + ck + 3dk + 2bk^2 + dk^2 + 2an + 2bn + 2cn + 3dn + 2akn + 2bkn + ckn + 2dkn + cn^2 + dn^2$$

```
CoefficientList[%, k]
```

$$\{2a + 2b + c + 2d + 2an + 2bn + 2cn + 3dn + cn^2 + dn^2,\ 2a + 4b + c + 3d + 2an + 2bn + cn + 2dn,\ 2b + d\}$$

```
Solve[{2 a + 2 b + c + 2 d + 2 a n + 2 b n + 2 c n + 3 d n + c n^2 + d n^2 == 0,
  2 a + 4 b + c + 3 d + 2 a n + 2 b n + c n + 2 d n == 0, 2 b + d == 0}, {a, b, c, d}]
```

$$\left\{\left\{a \to 0,\ c \to -d,\ b \to -\frac{d}{2}\right\}\right\}$$

The solutions to the linear system are parametrized by d. We can thus get a specific solution by setting $d = 1$. The resulting recurrence relation is as follows.

$$\frac{-1}{2}\, s[n+1,\ k] - s[n,\ k+1] + s[n+1,\ k+1] = 0 \tag{1}$$

for any integer j. So, when we sum both sides of the equation, we get

$$S(n+1) - s(n+1,\ n+1) + 2\,S(n) - 2\,s(n,\ 0) + 2\,s(n,\ n+1) - 2\,S(n+1) + 2\,s(n+1,\ 0) = 0$$

Or,

$$S(n+1) = 2\,S(n)$$

We solve it by iteratiom,

$$S(n+1) = 2\,S(n) = 4\,S(n-1) = 8\,S(n-2) = \ldots = 2^{n+1}\,S(0)$$

Since the base case is S(0) = 1, we conclude that

$$S(n) = 2^n$$

```
Sum[ Binomial[n + k, k] 2^(-k), {k, 0, n}]
```

2^n