15-354: CDM **K. Sutner**

Assignment 3 Due: **Sep. 22, 2023, 24:00**.

1. Semidecidable Sets and Computable Functions (40)

Background

We defined semidecidable sets as a generalization of decidable sets: on a Yes-instance the "semidecision algorithm" terminates, but on a No-instance it keeps running forever. There are many alternative characterizations that describe more directly the relationship between semidecidable sets and partial computable functions.

By an enumeration of $A \subseteq \mathbb{N}$ we mean a partial function $f : \mathbb{N} \nrightarrow \mathbb{N}$ so that the range of f is A. For simplicity, we will assume that the support of f is either all of $\mathbb{N}$ or some initial segment $\{0, 1, \ldots, n-1\}$. So

$$A = \{\, f(i) \mid i < N \,\} = f(0), f(1), f(2), \ldots$$

where $N = n$ or $N = \omega$. Note that we allow $n = 0$ corresponding to $A = \emptyset$. An enumeration is repetition-free if f is injective. A set is recursively enumerable (r.e.) if it can be enumerated by a computable function f.

Task

Assume that $A \subseteq \mathbb{N}$. Show the following.

A. All finite sets are recursively enumerable.

B. The set of primes is recursively enumerable.

C. The set of prime twins is recursively enumerable.

D. A is semidecidable iff it is recursively enumerable.

E. A is semidecidable iff it is recursively enumerable with a repetition-free enumeration.

F. Suppose A is infinite. Then A is decidable iff it is recursively enumerable with a strictly increasing enumeration.

Comment

Don't try to argue formally in terms of register machines, just use computability in the intuitive sense, much the way you would describe a solution to a problem in an algorithms class.

Note that it is currently unknown whether there are infinitely many prime twins—but that does not affect part (C).

2. The DASZ Operator (30)

Background

For this problem, consider non-decreasing lists of positive integers $A = (a_1, a_2, \ldots, a_w)$. We transform any such list into a new one according to the following simple recipe:

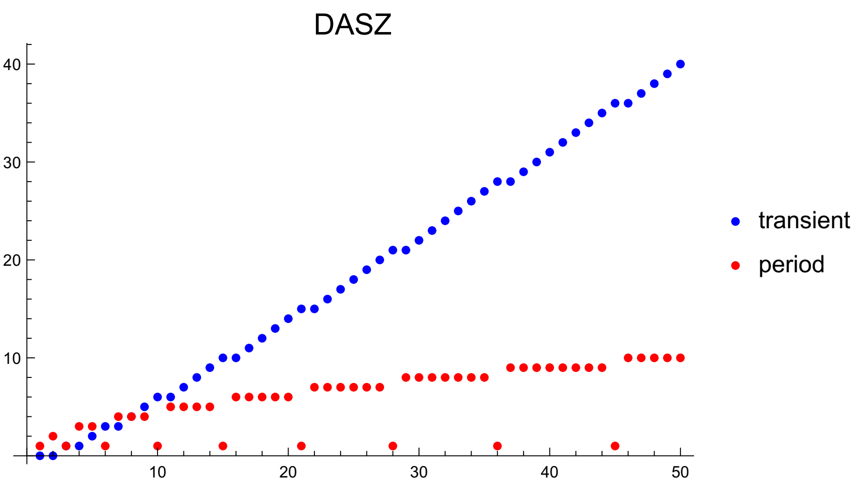
- Subtract 1 from all elements.
- Append the length of the list as a new element.
- Sort the list.
- Remove all 0 entries.

We will call this the DASZ operation (decrement, append, sort, kill zero) and write $D(A)$ for the new list (note that D really is a function). For example, $D(1, 3, 5) = (2, 3, 4)$, $D(4) = (1, 3)$ and $D(1, 1, 1, 1) = (4)$.

A single application of D is not too fascinating, but things become interesting when we iterate the operation: as it turns out, $D^t(A)$ always has a finite transient (and period), no matter how A is chosen. For example, the transient and period of $(1, 1, 1, 1, 1)$ are both 3:

	0	1	1	1	1	1
	1	5				
	2	1	4			
transient	3	2	3			
	4	1	2	2		
	5	1	1	3		
period	6	2	3			

Here is a plot of the transients and periods of all starting lists $A = (n)$ for $n \leq 50$.



Note the fixed points $D(A) = A$, the few red dots at the bottom.

Task

A. Show that all transients must be finite.

B. Characterize all the fixed points of the DASZ operation.

C. Determine which initial lists $A = (n)$ lead to a fixed point.

3. Speeding Up Iteration (30)

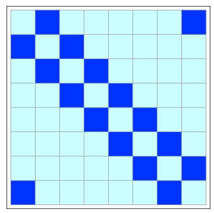
Background

The method of fast exponentiation can sometimes be used to speed-up the computation of $f^t(a)$ for some endofunction $f : A \to A$. Here is an example, and a limitation to this speed-up effect.

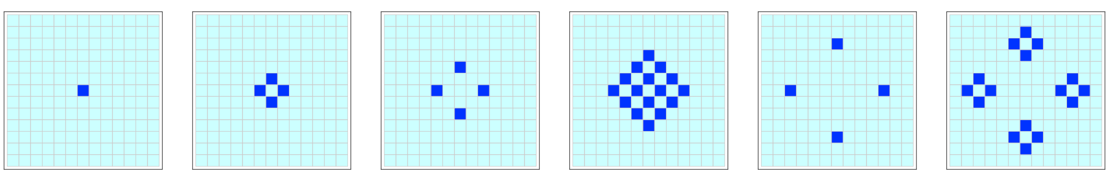
For $n \geq 1$ let $A = \mathbf{2}^{n \times n}$ be the set of all $n \times n$ Boolean matrices. Define the circulant matrix C by

$$C(i,j) = \begin{cases} 1 & \text{if } j = i \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

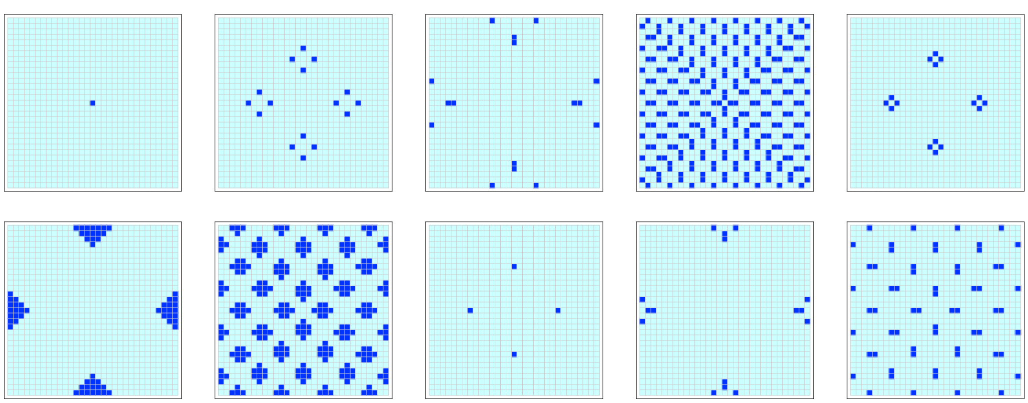
Here the indices are supposed to wrap around, so that, say, C_8 has the form



Lastly, define $f : A \to A$ by $f(X) = C \cdot X + X \cdot C$ where for the matrix multiplication we interpret addition as logical *exclusive or* and multiplication as logical *and*. Here is the effect of applying f^t to the 13×13 matrix with a single 1 in the center, rest all 0's, for $t = 0, 1, \ldots, 5$.



Note how, at times 2 and 3, 4 and 5, the pictures contain 4 copies of the pictures at times 0 and 1. Similarly, the effect of f^t on the 31×31 single-point matrix, for times $t = 0, 10, 20, \ldots, 90$.



The patterns are rather surprising, you might want to write a program that the produces the whole orbit (and try different matrix sizes).

Task

A. Describe the effect of f on $X \in A$ in geometric terms.

B. Show how to compute $f^t(X)$ for $X \in A$ in time $O(\mathsf{pol}(n) \log t)$ where pol is a low-degree polynomial depending only on n. Make sure to explain the degree of pol.

   Hint: express f as a single matrix multiplication. You might want to look up Kronecker product.

C. Show that $\mathbb{P} = \mathbb{NP}$ if exponential speed-up is always possible.

Comment

For part (C), find a way to determine satisfiability of a Boolen formula $\phi(x_1, \ldots, x_n)$ by iterating a function f defined essentially on $\mathbf{2}^n$.