15-354: CDM **K. Sutner**

Assignment 1 Due: **Sep. 8, 2023**.

1. Primitive Recursion (50)

Background

Recall that bounded search is defined as follows. Let $g : \mathbb{N}^{n+1} \to \mathbb{N}$. Then $f = \mathsf{BS}[g] : \mathbb{N}^{n+1} \to \mathbb{N}$ is the function defined by

$$f(x, \boldsymbol{y}) = \begin{cases} \min\big(z < x \mid g(z, \boldsymbol{y}) = 0 \big) & \text{if } z \text{ exists,} \\ x & \text{otherwise.} \end{cases}$$

Task

A. Show that $f(x, \boldsymbol{y}) = \sum_{z<h(x)} g(z, \boldsymbol{y})$ is primitive recursive when h is primitive recursive and strictly monotonic.

B. What if h is not monotonic?

C. Show that $\mathsf{BS}[g]$ is primitive recursive whenever g is.

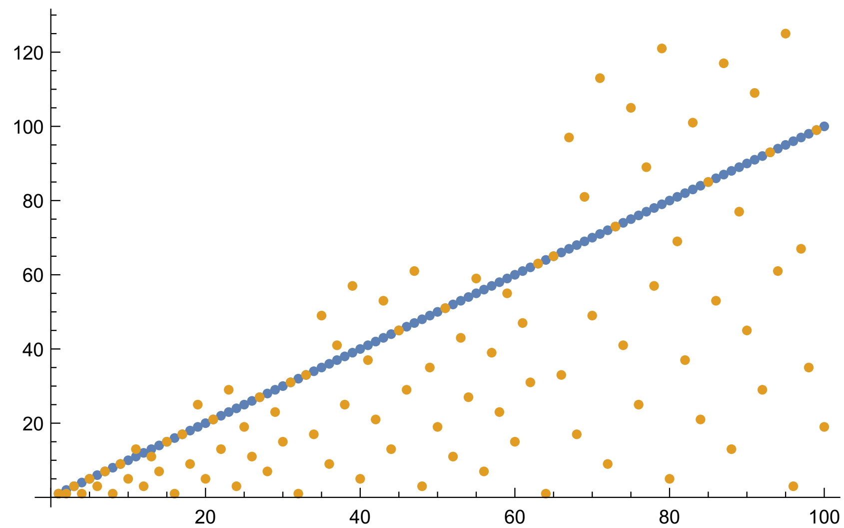
2. A Recursion (50)

Background

Consider the following function f, presumably defined on the positive integers.

$$
\begin{aligned}
f(1) &= 1 \\
f(3) &= 3 \\
f(2n) &= f(n) \\
f(4n+1) &= 2f(2n+1) - f(n) \\
f(4n+3) &= 3f(2n+1) - 2f(n)
\end{aligned}
$$

For what it's worth, here is a plot of the first few values.



Task

A. Consider small values of f and conjecture an explicit, non-recursive definition of f.

B. Prove that your definition is correct and conclude that f is indeed a function from $\mathbb{N}_+$ to $\mathbb{N}_+$.

C. Is f primitive recursive?

Comment

Implement f and experiment.

3. Primitive Recursive Word Functions (40)

Background

We defined primitive recursive functions on the naturals. A similar definition would also work for words over some alphabet Σ. We write ε for the empty word and $\Sigma^\star$ for the set of all words over Σ. Consider the clone of word functions generated by the basic functions

- Constant empty word $E : (\Sigma^\star)^0 \to \Sigma^\star$, $E() = \varepsilon$,
- Append functions $S_a : \Sigma^\star \to \Sigma^\star$, $S(x) = x\,a$ where $a \in \Sigma$.

and closed under primitive recursion over words: suppose we have a function $g : (\Sigma^\star)^n \to \Sigma^\star$ and a family of functions $h_a : (\Sigma^\star)^{n+2} \to \Sigma^\star$, where $a \in \Sigma$. We can then define a new function $f : (\Sigma^\star)^{n+1} \to \Sigma^\star$ by

$$\begin{aligned} f(\varepsilon, \boldsymbol{y}) &= g(\boldsymbol{y}) \\ f(xa, \boldsymbol{y}) &= h_a(x, f(x, \boldsymbol{y}), \boldsymbol{y}) \qquad a \in \Sigma \end{aligned}$$

We will call the members of this clone the word primitive recursive (w.p.r.) functions.

Task

1. Show that the reversal operation $\mathsf{rev}(x) = x_n x_{n-1} \dots x_1$ is w.p.r.
2. Show that the prepend operations $\mathsf{pre}_a(x) = a\,x$ are w.p.r.
3. Show that the concatenation operation $\mathsf{cat}(x, y) = x\,y$ is w.p.r.
4. Prove that every primitive recursive function is also a word primitive recursive function. By this we mean that for every p.r. function $f : \mathbb{N}^k \to \mathbb{N}$ there is a w.p.r. function $F : (\Sigma^\star)^k \to \Sigma^\star$ so that $f(\boldsymbol{x}) = D(F(C(\boldsymbol{x})))$ where C and D are simple coding and decoding functions (between numbers and words).
5. Prove the opposite direction: every w.p.r. is already p.r., using coding and decoding as in the last problem.

Comment

For the last part, don't get bogged down in tons of technical details, just explain how one would go about proving this.