Constructive Logic (15-317), Fall 2012
Assignment 5: Classical Logic

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Due: Thursday, October 18, 2012 (before class)

In this assignment, you will investigate the relationship between constructive and classical logic.

1 Double-Negation Translation (12 points)

Unlike in constructive logic, which has only a judgment for truth, classical logic has judgments for truth, falsity, and contradiction. Classical logic also has a primitive notion of negation ($\neg A$), whereas constructive logic simply defines $\neg A$ to be $A \supset \bot$.

The Gödel–Gentzen double–negation translation takes a classical proposition A to a constructive proposition A^*, and is defined inductively on the structure of A as follows:

$$\begin{aligned}
\top^* &= \top \\
\bot^* &= \bot \\
(A \wedge B)^* &= A^* \wedge B^* \\
(A \supset B)^* &= A^* \supset B^* \\
(A \vee B)^* &= \neg\neg(A^* \vee B^*) \\
(\neg A)^* &= \neg A^* \\
P^* &= \neg\neg P \text{ where } P \text{ atomic}
\end{aligned}$$

(*Note:* On the left, $\neg$ is the primitive classical notion of negation; on the right, $\neg$ is an abbreviation for $- \supset \bot$.)

Task 1 (12 points)**.** Prove that, for any classical proposition A,

$$\cdot \vdash \neg\neg A^* \supset A^* \ \mathit{true}$$

is derivable (constructively). You need only show the cases for:

- $\top$
- atomic propositions
- $\supset$
- $\vee$

2 Embedding Classical Logic (28 points)

The Gödel–Gentzen double–negation translation allows us to *embed* classical logic into constructive logic in the following sense:

Theorem 1.

1. *If* $\Gamma \vdash_C A$ *true, then* $\Gamma^* \vdash A^*$ *true.*
2. *If* $\Gamma \vdash_C A$ *false, then* $\Gamma^* \vdash \neg A^*$ *true.*
3. *If* $\Gamma \vdash_C \#$*, then* $\Gamma^* \vdash \bot$ *true.*

In the above theorem, Γ^* is the result of applying the double–negation translation to each proposition in the context; $\vdash_C$ indicates a classical derivation, using the rules listed at the end of this assignment; and $\vdash$ indicates a constructive derivation. In other words, this theorem states that any classically–derivable proposition has a constructively–derivable counterpart, given by the translation.

Task 2 (28 points)**.** Prove Theorem 1, showing the following cases:

- $\supset T$
- $\vee T1$
- $\supset F$
- $\vee F$
- $\neg F$
- $\#$
- $\mathsf{PBC}T$

You may use the result stated in Task 1. (*Hint:* You will also need weakening.)

A Classical rules

$$\frac{\Gamma, x : A\ true \vdash_C M : B\ true}{\Gamma \vdash_C \lambda x : A.M : A \supset B\ true} \supset T \qquad \frac{\Gamma \vdash_C M : A\ true \quad \Gamma \vdash_C N : B\ true}{\Gamma \vdash_C \langle M, N \rangle : A \wedge B\ true} \wedge T \qquad \frac{}{\Gamma \vdash_C \star : \top\ true} \top T$$

$$\frac{\Gamma \vdash_C M : A\ true}{\Gamma \vdash_C \mathsf{in}_1\ M : A \vee B\ true} \vee T1 \qquad \frac{\Gamma \vdash_C M : B\ true}{\Gamma \vdash_C \mathsf{in}_2\ M : A \vee B\ true} \vee T2 \qquad \frac{\Gamma \vdash_C K : A\ false}{\Gamma \vdash_C \neg K : \neg A\ true} \neg T$$

$$\frac{\Gamma \vdash_C M : A\ true \quad \Gamma \vdash_C K : B\ false}{\Gamma \vdash_C M; K : A \supset B\ false} \supset F \qquad \frac{\Gamma \vdash_C K : A\ false \quad \Gamma \vdash_C L : B\ false}{\Gamma \vdash_C [K, L] : A \vee B\ false} \vee F \qquad \frac{}{\Gamma \vdash_C \bullet : \bot\ false} \bot F$$

$$\frac{\Gamma \vdash_C K : A\ false}{\Gamma \vdash_C \pi_1 \cdot K : A \wedge B\ false} \wedge F1 \qquad \frac{\Gamma \vdash_C K : B\ false}{\Gamma \vdash_C \pi_2 \cdot K : A \wedge B\ false} \wedge F2 \qquad \frac{\Gamma \vdash_C M : A\ true}{\Gamma \vdash_C \neg M : \neg A\ false} \neg F$$

$$\frac{\Gamma \vdash_C M : A\ true \quad \Gamma \vdash_C K : A\ false}{\Gamma \vdash_C \langle M \rhd K \rangle : \#} \# \qquad \frac{}{\Gamma, x : A\ true \vdash_C x : A\ true} \mathsf{hyp}T \qquad \frac{}{\Gamma, x : A\ false \vdash_C x : A\ false} \mathsf{hyp}F$$

$$\frac{\Gamma, u : A\ false \vdash_C E : \#}{\Gamma \vdash_C u : A\ false.E : A\ true} \mathsf{PBC}T \qquad \frac{\Gamma, u : A\ true \vdash_C E : \#}{\Gamma \vdash_C u : A\ true.E : A\ false} \mathsf{PBC}F$$