Constructive Logic (15-317), Fall 2014
Assignment 10: Linear Logic

Joe Tassarotti (`jtassaro@andrew`), Evan Cavallo (`ecavallo@andrew`)

Out: Wednesday, November 26, 2014
Due: Thursday, December 4, 2014 (before class)

In the final assignment, you will explore linear logic. First, you will do some derivations in linear logic. Then, you will have to show local soundness and completeness for parts of linear logc. Finally, you'll work through one way to interpret linear logic, and use that to prove that certain judgments are unprovable.

Your work should be submitted electronically before the beginning of the class. Please convert your homework to a PDF file titled `hw10.pdf`, and put the file in

`/afs/andrew/course/15/317/submit/<your andrew id>`

If you are familiar with LaTeX, you are encouraged to use this document as a template for typesetting your solutions, but you may alternatively write your solutions *neatly* by hand and scan them.

1 A Few Final Truths (15 points)

Task 1 (12 pts)**.** For each of the following judgments, say whether they hold or not. If they hold, give a derivation.

1. $A \multimap (B \otimes C) \Vdash (A \multimap B) \otimes (A \multimap C)$
2. $A \multimap (B \mathbin{\&} C) \Vdash (A \multimap B) \mathbin{\&} (A \multimap C)$
3. $(A \multimap C) \oplus (B \multimap C) \Vdash (A \otimes B) \multimap C$
4. $A \multimap \mathbf{1}, (A \otimes A) \Vdash A$

Task 2 (3pts)**.** Give a focused proof of the following, where P, Q, and R are atomic propositions:

$$\longrightarrow (\downarrow((\downarrow P \oplus \downarrow Q) \multimap R) \otimes \downarrow P) \multimap R$$

2 A Harmonious End (8 points)

Task 3 (4 pts)**.** Show that the $\multimap$ connective is locally sound and complete.

Task 4 (4 pts)**.** Show that the $\&$ connective is locally sound and complete.

3 Model Theory (23 points)

It turns out that we can interpret the propositions of linear logic in terms of operations on sets of natural numbers. In this section, we will prove that this interpretation works, and use it show that certain judgments are unprovable.

Let f be some function which maps atomic propositions to sets of natural numbers. Then we can extend this to a function $[\![\cdot]\!]_f$ which maps propositions in linear logic into sets of natural numbers, as follows:

$$\begin{aligned}
[\![P]\!]_f &= f(P) \qquad \text{for } P \text{ atomic} \\
[\![X \oplus Y]\!]_f &= [\![X]\!]_f \cup [\![Y]\!]_f \\
[\![X \& Y]\!]_f &= [\![X]\!]_f \cap [\![Y]\!]_f \\
[\![X \multimap Y]\!]_f &= \{z \mid \forall x \in [\![X]\!]_f .\, x + z \in [\![Y]\!]_f\} \\
[\![X \otimes Y]\!]_f &= \{x + y \mid x \in [\![X]\!]_f, y \in [\![Y]\!]_f\} \\
[\![\mathbf{1}]\!]_f &= \{0\} \\
[\![\mathbf{0}]\!]_f &= \emptyset \\
[\![\top]\!]_f &= \mathbb{N}
\end{aligned}$$

Suppose $\Gamma = A_1, \ldots, A_n$. Write $\Gamma^{\otimes}$ for the formula $A_1 \otimes \ldots \otimes A_n$. (If $\Gamma = \cdot$, then $\Gamma^{\otimes} = \mathbf{1}$). Then, the above intepretation has the following property:

Theorem 1. *If $\Gamma \Vdash A$, then $0 \in [\![\Gamma^{\otimes} \multimap A]\!]_f$.*

Task 5 (15 pts)**.** Prove theorem 1 by induction on the derivation in linear logic. You only have to do the cases for:

- $\multimap E$

- $\otimes I$

- $\oplus E$

We can use theorem 1 as a way to prove that certain judgments are unprovable. This is an alternative to the proofs we've seen earlier in the class that used sequent calculus to make similar arguments.

Task 6 (8 pts)**.** Use theorem 1 to show that the following judgments are *not* provable, in general. In each example, P and Q are atomic propositions. (Hint: Come up with an appropriate function f and then show that 0 is not in the set we get when we apply $[\![\cdot]\!]_f$.).

- $\cdot \Vdash P \oplus (P \multimap \mathbf{0})$

- $\cdot \Vdash (P \otimes Q) \multimap (P \mathbin{\&} Q)$