Constructive Logic (15-317), Fall 2010
Assignment 5: Classical Logic and Inversion
Solution

Chris Martens (cmartens@cs)

Out: Thursday, October 7, 2010
Due: Thursday, October 14, 2010 (at the end of class)

In this assignment, you will use classical logic as a setting for proofs and computations. You will also practice doing sequent calculus proofs in a more restricted setting.

Your work should be submitted at the beginning of class. If you are familiar with LaTeX, you are encouraged to use this document as a template for typesetting your solutions, but you may alternatively write your solutions *neatly* by hand.

1 Classical Proofs and Computations (18 points)

Task 1 (9 points)**.** Prove the following classical propositions using the calculus given in class.

- $(A \supset B) \vee (B \supset A)$

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A\ true}\,x}{B \supset A\ true}\supset T^y}{(A \supset B) \vee (B \supset A)\ true}\vee T_2 \quad \dfrac{}{(A \supset B) \vee (B \supset A)\ false}\,u}{\#}\,contra}{B\ true}\,T\#^v}{A \supset B\ true}\supset T^x}{(A \supset B) \vee (B \supset A)\ true}\vee T_1 \quad \dfrac{}{(A \supset B) \vee (B \supset A)\ false}\,u}{\dfrac{\#}{(A \supset B) \vee (B \supset A)\ true}\,T\#^u}\,contra$$

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- $((A \supset B) \supset A) \supset A$

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A\supset B)\supset A\ \mathit{true}}\,x \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A\ \mathit{true}}\,y \quad \dfrac{}{A\ \mathit{false}}\,a}{\#}\,\mathit{contra}}{B\ \mathit{true}}\,T\#^{b}}{A\supset B\ \mathit{true}}\,{\supset}T^{y} \qquad \dfrac{}{A\ \mathit{false}}\,u}{(A\supset B)\supset A\ \mathit{false}}\,{\supset}F}{\#}\,\mathit{contra}}{A\ \mathit{true}}\,T\#^{u}}{((A\supset B)\supset A)\supset A\ \mathit{true}}\,{\supset}T^{x}$$

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- $(A\supset B)\wedge(\neg A\supset B)\supset B$

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A\supset B)\wedge(\neg A\supset B)\ \mathit{true}}\,x \qquad \dfrac{\dfrac{\overset{\mathcal{D}}{A\ \mathit{true}} \quad \dfrac{}{B\ \mathit{false}}\,b}{A\supset B\ \mathit{false}}\,{\supset}F}{(A\supset B)\wedge(\neg A\supset B)\ \mathit{false}}\,{\wedge}F_1}{\#}\,\mathit{contra}}{B\ \mathit{true}}\,T\#^{b}}{(A\supset B)\wedge(\neg A\supset B)\supset B\ \mathit{true}}\,{\supset}T^{x}$$

where $\mathcal{D}$ is

$$\dfrac{\dfrac{\dfrac{}{(A\supset B)\wedge(\neg A\supset B)\ \mathit{true}}\,x \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A\ \mathit{false}}\,a}{\neg A\ \mathit{true}}\,\neg T \quad \dfrac{}{B\ \mathit{false}}\,b}{\neg A\supset B\ \mathit{false}}\,{\supset}F}{(A\supset B)\wedge(\neg A\supset B)\ \mathit{false}}\,{\wedge}F_2}{\#}\,\mathit{contra}}{A\ \mathit{true}}\,T\#^{a}$$

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Task 2 (9 points)**.** Give proof terms corresponding to your proofs of the above.

Solution. We will annotate our letcc and frame expressions with the type of their assumptions, using the notation $x : A$ to stand for proof assumptions $x : A\ \mathit{true}$ and $a : \overline{A}$ to stand for refutation assumptions $a : A\ \mathit{false}$.

1. letcc $u{:}\overline{(A \supset B) \vee (B \supset A)}.(\mathsf{in}_1\ (\lambda x{:}A.\mathsf{letcc}\ b{:}\overline{B}.(\mathsf{in}_2\ (\lambda y{:}B.x)) \rhd u)) \rhd u$

2. $\lambda x{:}(A \supset B) \supset A.\mathsf{letcc}\ a{:}\overline{A}.x \rhd ((\lambda y{:}A.\mathsf{letcc}\ b{:}\overline{B}.y \rhd a); a)$

3. $\lambda x{:}(A \supset B) \wedge (\neg A \supset B).\mathsf{letcc}\ b{:}\overline{B}.x \rhd \rho_1((\mathsf{letcc}\ a{:}\overline{A}.x \rhd \rho_2(\neg a; b)); b)$

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1.1 Extra Credit

The SML/NJ programming language (smlnj.org) has a continuation library that matches the following signature:

```
signature CONT = sig
  val callcc : ('a cont -> 'a) -> 'a
  val throw : 'a cont -> 'a -> 'b
end
```

Roughly, these correspond to our letcc and $M \rhd K$ constructs. `'a cont` can be thought of as $\neg A$.

You can open by running `sml` at the prompt on any Andrew Unix machine and typing

```
- open SMLofNJ.Cont;
```

(This will also give you access to several other primitives; you should ignore them for the sake of this exercise.)

Extra Credit Task 1 (10 points)**.** Code up SML expressions inhabiting types that correspond to the above propositions using `callcc` and `throw`. If you make any nonobvious representation choices, document them. (You should also stay within the pure fragment of ML, which is to say: don't cheat by using references or exceptions.)

2 Preservation of Contradiction (12 points)

The statement of preservation for the classical computation system is

If $S \longrightarrow S'$ and $S : \#$ then $S' : \#$.

The proof proceeds by induction on the derivation of $S \longrightarrow S'$.

Here is an example case:

Suppose $\langle M, N\rangle \rhd \rho_1 K \longrightarrow M \rhd K$.

By our other assumption, we have

$$\langle M, N\rangle \rhd \rho_1 K : \#$$

By inversion, this gives us

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ M : A\ \mathit{true}\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ N : B\ \mathit{true}\end{array}}{\langle M, N\rangle : A \wedge B\ \mathit{true}}\ \wedge T \qquad \dfrac{\begin{array}{c}\mathcal{E}\\ K : A\ \mathit{false}\end{array}}{\rho_1 K : A \wedge B\ \mathit{false}}\ \wedge F}{\langle M, N\rangle \rhd \rho_1 K : \#}\ \mathit{contra}$$

In this case, $S' = M \rhd K$, which we need to show is a proof of #. This follows by *contra* on preceding derivations:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ M : A\ \mathit{true}\end{array} \quad \begin{array}{c}\mathcal{E}\\ K : A\ \mathit{false}\end{array}}{M \rhd K : \#}\ \mathit{contra}$$

Recall that in the subject reduction proof for intuitionistic natural deduction, we needed the following substitution principle:

If $\begin{array}{c}[x : A]\\ N : B\end{array}$ *and* $M : A$ *then* $[M/x]N : B$.

To prove the remaining cases, we need several variations on this principle:

Substitution Principles for Classical Computation:

1. *If* $\begin{array}{c}[x : A\ \mathit{true}]\\ N : B\ \mathit{true}\end{array}$ *and* $M : A\ \mathit{true}$ *then* $[M/x]N : B\ \mathit{true}$.
2. *If* $\begin{array}{c}[x : A\ \mathit{true}]\\ K : B\ \mathit{false}\end{array}$ *and* $M : A\ \mathit{true}$ *then* $[M/x]K : B\ \mathit{false}$.
3. *If* $\begin{array}{c}[x : A\ \mathit{true}]\\ S : \#\end{array}$ *and* $M : A\ \mathit{true}$ *then* $[M/x]S : \#$.
4. *If* $\begin{array}{c}[a : A\ \mathit{false}]\\ N : B\ \mathit{true}\end{array}$ *and* $K : A\ \mathit{false}$ *then* $[K/a]N : B\ \mathit{true}$.
5. *If* $\begin{array}{c}[a : A\ \mathit{false}]\\ L : B\ \mathit{false}\end{array}$ *and* $K : A\ \mathit{false}$ *then* $[K/a]L : B\ \mathit{false}$.
6. *If* $\begin{array}{c}[a : A\ \mathit{false}]\\ S : \#\end{array}$ *and* $K : A\ \mathit{false}$ *then* $[K/a]S : \#$.

To formally finish this proof, we would need to define all of these forms of substitution and prove each of these properties. For the sake of this assignment, you may assume them as principles—just take care to state where you make use of them.

Task 3 (12 points)**.** Prove the remaining cases of subject reduction.

Solution.

- Case:

  Suppose $\langle M, N\rangle \triangleright \rho_2 K \longrightarrow N \triangleright K$.

  By our other assumption, we have

  $$\langle M, N\rangle \triangleright \rho_2 K : \#$$

  By inversion,

  $$\dfrac{\dfrac{\overset{\mathcal{D}_1}{M : A\ \textit{true}} \quad \overset{\mathcal{D}_2}{N : B\ \textit{true}}}{\langle M, N\rangle : A \wedge B\ \textit{true}} \wedge T \qquad \dfrac{\overset{\mathcal{E}}{K : B\ \textit{false}}}{\rho_2 K : A \wedge B\ \textit{false}} \wedge F}{\langle M, N\rangle \triangleright \rho_2 K : \#}\ \textit{contra}$$

  To show: $N \triangleright K : \#$.

  $$\dfrac{\overset{\mathcal{D}_2}{N : B\ \textit{true}} \quad \overset{\mathcal{E}}{K : B\ \textit{false}}}{N \triangleright K : \#}\ \textit{contra}$$

- Case:

  Suppose $\mathsf{in}_1\ M \triangleright [K_1, K_2] \longrightarrow M \triangleright K_1$.

  By our other assumption, we have

  $$\mathsf{in}_1\ M \triangleright [K_1, K_2] : \#$$

  By inversion,

  $$\dfrac{\dfrac{\overset{\mathcal{D}}{M : A\ \textit{true}}}{\mathsf{in}_1\ M : A \vee B\ \textit{true}} \vee T_1 \qquad \dfrac{\overset{\mathcal{E}_1}{K_1 : A\ \textit{false}} \quad \overset{\mathcal{E}_2}{k_2 : B\ \textit{false}}}{[K_1, K_2] : A \vee B\ \textit{false}} \vee F}{\mathsf{in}_1\ M \triangleright [K_1, K_2] : \#}\ \textit{contra}$$

  To show: $M \triangleright K_1 : \#$.

  $$\dfrac{\overset{\mathcal{D}_2}{M : A\ \textit{true}} \quad \overset{\mathcal{E}_1}{K_1 : A\ \textit{false}}}{M \triangleright K_1 : \#}\ \textit{contra}$$

- Case:

  Suppose $\mathsf{in}_2\ M \triangleright [K_1, K_2] \longrightarrow M \triangleright K_2$.

  This case is symmetric to the previous one.

- Case:

  Suppose $\lambda x{:}A.M \rhd (N;K) \longrightarrow [N/x]M \rhd K$.

  By our other assumption, we have

$$\frac{\lambda x{:}A.M : A \supset B\ \mathit{true} \quad (N;K) : A \supset B\ \mathit{false}}{\lambda x{:}A.M \rhd (N;K) : \#}\ \mathit{contra}$$

  By inversion,

$$\frac{\dfrac{\begin{array}{c}[x : A\ \mathit{true}]\\ \mathcal{D}\\ M : B\ \mathit{true}\end{array}}{\lambda x{:}A.M : A \supset B\ \mathit{true}} \supset T^x \quad \dfrac{\begin{array}{c}\mathcal{E}_1\\ N : A\ \mathit{true}\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ K : B\ \mathit{false}\end{array}}{(N;K) : A \supset B\ \mathit{true}} \supset F}{\lambda x{:}A.M \rhd (N;K) : \#}\ \mathit{contra}$$

  By our substitution principle, $[\mathcal{E}_1/x]\mathcal{D} :: [N/x]M : B\ \mathit{true}$.

$$\frac{\begin{array}{c}[\mathcal{E}_1/x]\mathcal{D}\\ [N/x]M : B\ \mathit{true}\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ K : B\ \mathit{false}\end{array}}{[N/x]M \rhd K : \#}\ \mathit{contra}$$

- Suppose $\neg K \rhd \neg M \longrightarrow M \rhd K$.

  By our other assumption,

$$\frac{\neg K : \neg A\ \mathit{true} \quad \neg M : \neg A\ \mathit{false}}{\neg K \rhd \neg M : \#}\ \mathit{contra}$$

  By inversion,

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}\\ K : A\ \mathit{false}\end{array}}{\neg K : \neg A\ \mathit{true}} \neg T \quad \dfrac{\begin{array}{c}\mathcal{E}\\ M : A\ \mathit{true}\end{array}}{\neg M : \neg A\ \mathit{false}} \neg F}{\neg K \rhd \neg M : \#}\ \mathit{contra}$$

  So

$$\frac{M : A\ \mathit{true} \quad K : A\ \mathit{false}}{M \rhd K : \#}\ \mathit{contra}$$

  as desired.

- Suppose $M \rhd \textsf{frame}\ x.S \longrightarrow [M/x]S$.

  By our other assumption and inversion,

$$\dfrac{\begin{array}{c}\mathcal{D}\\ M : A\ \mathit{true}\end{array} \quad \dfrac{\begin{array}{c}[x : A\ \mathit{true}]\\ \mathcal{E}\\ S : \#\end{array}}{\textsf{frame}\ x.S : A\ \mathit{false}}\ F\#^x}{M \rhd \textsf{frame}\ x.S : \#}\ \mathit{contra}$$

  By substitution principle 3, $[M/x]S : \#$, which is what we needed to show.

- Suppose $\textsf{letcc}\ a.S \rhd K \longrightarrow [K/a]S$.

  By our other assumption and inversion,

$$\dfrac{\dfrac{\begin{array}{c}[a : A\ \mathit{false}]\\ \mathcal{D}\\ S : \#\end{array}}{\textsf{letcc}\ x.S : A\ \mathit{true}}\ T\#^a \quad \begin{array}{c}\mathcal{E}\\ K : A\ \mathit{false}\end{array}}{\textsf{letcc}\ x.S \rhd K : \#}\ \mathit{contra}$$

  By substitution principle 6, $[K/a]S : \#$, which is what we needed to show.

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3 Inversion (10 points)

Remember that in the $\supset L$ rule for the inversion sequent calculus, we kept the $A \supset B$ hypothesis in the first premise but not the second:

$$\dfrac{\Gamma, \boxed{A \supset B} \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ \supset L$$

Claim: a proof of the following sequent requires us to make use of that persistent assumption.

$$(P \vee (P \supset Q)) \supset Q \longrightarrow Q$$

Task 4 (5 points)**.** Prove the sequent in the inversion calculus.

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \vee (P \supset Q)) \supset Q, P \longrightarrow P}\,\mathit{init}}{(P \vee (P \supset Q)) \supset Q, P \longrightarrow P \vee (P \supset Q)}\,\vee R_1 \quad \dfrac{}{(P \vee (P \supset Q)) \supset Q, P, Q \longrightarrow Q}\,\mathit{init}}{(P \vee (P \supset Q)) \supset Q, P \longrightarrow Q}\,\supset L}{(P \vee (P \supset Q)) \supset Q \longrightarrow P \supset Q}\,\supset R}{(P \vee (P \supset Q)) \supset Q \longrightarrow P \vee (P \supset Q)}\,\vee R_2 \quad \dfrac{}{(P \vee (P \supset Q)) \supset Q, Q \longrightarrow Q}\,\mathit{init}}{(P \vee (P \supset Q)) \supset Q \longrightarrow Q}\,\supset L$$

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Task 5 (5 points)**.** Prove that the sequent is unprovable if we take out the $\boxed{A \supset B}$ assumption from the first premise of $\supset L$.

Solution. The only rule that applies at first is $\supset L$:

$$\dfrac{\cdot \longrightarrow P \vee (P \supset Q) \qquad (P \vee (P \supset Q)) \supset Q, Q \longrightarrow Q}{(P \vee (P \supset Q)) \supset Q \longrightarrow Q}\,\supset L$$

It suffices to show that the left premise is unprovable.
Two rules apply: $\vee R_1$ and $\vee R_2$.
Case:

$$\dfrac{\cdot \longrightarrow P}{\cdot \longrightarrow P \vee (P \supset Q)}\,\vee R_1$$

No rules apply.
Case:

$$\dfrac{\dfrac{P \longrightarrow Q}{\cdot \longrightarrow P \supset Q}\,\supset R}{\cdot \longrightarrow P \vee (P \supset Q)}\,\vee R_2$$

$\supset R$ is the only rule that applies, and then no rules apply. ■