Constructive Logic (15-317), Fall 2014
Assignment 5: Double-Negation Translation

Joe Tassarotti (`jtassaro@andrew`), Evan Cavallo (`ecavallo@andrew`)

Out: Thursday, October 9, 2014
Due: October 16, 2014 (before class)

In this assignment, you will practice classical logic with proof terms and explore translations from classical logic into constructive logic.

Your work should be submitted electronically before the beginning of class. Please convert your homework to a PDF file titled `hw05.pdf`, and put the file in:

```
/afs/andrew/course/15/317/submit/<your andrew id>
```

If you are familiar with LaTeX, you are encouraged to use this document as a template for typesetting your solutions, but you may alternatively write your solutions *neatly* by hand and scan them.

1 Classical Proofs (9 points)

Task 1 (4 points). Give a proof of $\neg(\neg A \wedge \neg B) \supset (A \vee B)$ *true* in classical logic. Use only the base rules of classical logic – not, for example, the derived elimination rules we discussed in class.

Task 2 (2 points). Write a proof term for the proof you gave in Task 1.

Task 3 (3 points). Assume you have a term $a : A$ *true* and a continuation $k_{init} : A \vee B$ *false*. Show a step-by-step reduction of the state obtained by feeding your proof term from Task 2 into the continuation $\neg(\pi_1 \cdot \neg a); k_{init} : \neg(\neg A \wedge \neg B) \supset (A \vee B)$ *false*.

2 Double-Negation Translation (27 points)

In class we showed that it was possible to embed classical logic into constructive logic via a double-negation translation. In this section, we will explore a different translation that inserts fewer negations. The Gödel–Gentzen double–negation translation takes a classical proposition A to a constructive proposition A^*, and is defined inductively on the structure of A as follows:

$$
\begin{aligned}
\top^* &= \top \\
\bot^* &= \bot \\
(A \wedge B)^* &= A^* \wedge B^* \\
(A \supset B)^* &= A^* \supset B^* \\
(A \vee B)^* &= \neg\neg(A^* \vee B^*) \\
(\neg A)^* &= \neg A^* \\
P^* &= \neg\neg P \text{ where } P \text{ atomic}
\end{aligned}
$$

(*Note:* On the left, $\neg$ is the primitive classical notion of negation; on the right, $\neg$ is an abbreviation for $- \supset \bot$. Observe that here we only insert negations for disjunctions and atomics, unlike in the Kolmogorov version done in class.)

Just like the Kolmogorov translation, Gödel–Gentzen double–negation also translation allows us to embed classical logic into constructive logic in the following sense:

Theorem 1.

1. *If* $\Gamma \vdash_C A$ *true, then* $\Gamma^* \vdash A^*$ *true.*
2. *If* $\Gamma \vdash_C A$ *false, then* $\Gamma^* \vdash \neg A^*$ *true.*
3. *If* $\Gamma \vdash_C \#$, *then* $\Gamma^* \vdash \bot$ *true.*

In the above theorem, Γ^* is the result of applying the double–negation translation to each proposition in the context: a hypothesis A *true* becomes A^* *true*, and a hypothesis A *false* becomes $\neg A^*$ *true*. $\vdash_C$ indicates a classical derivation, using the rules listed at the end of this assignment, and $\vdash$ indicates a constructive derivation. In other words, this theorem states that any classically–derivable proposition has a constructively–derivable counterpart, given by the translation.

Task 4 (12 points)**.** Prove that, for any classical proposition A,

$$\cdot \vdash \neg\neg A^* \supset A^* \textit{ true}$$

is derivable (constructively). You need only show the cases for:

- $\top$
- $\supset$
- $\vee$

Task 5 (15 points)**.** Prove Theorem 1, showing the following cases:

- $\supset T$
- $\vee T1$
- $\mathsf{PBC}T$

You may use the result stated in Task 4. (*Hint:* You will also need weakening.)

3 Friedman's Famous Trick (10 points)

In class, we proved Friedman's theorem, which shows that classical proofs of certain propositions can be converted into constructive proofs of the same proposition. Applying the double-negation translation to a proposition A produces a proposition A^* which is provable in natural deduction without the use of the $\bot E$ rule. If we replace all instances of $\bot$ in the proof with a proposition φ, we obtain a proposition $[\varphi/\bot]A$ which is also constructively provable. Friedman's trick consists in choosing φ carefully so that a A is constructively derivable from $[\varphi/\bot]A^*$.

Task 6 (10 points)**.** Assume you have a classical proof of $P \wedge Q$ *true* where P and Q are atomic. Use Friedman's trick to obtain a constructive proof of $P \wedge Q$ *true* from the proof of $(P \wedge Q)^*$ *true* provided by the double negation translation we defined in class. (*Hint:* Use the trick twice to prove P *true* and Q *true* separately.)

A Classical rules

$$\frac{\Gamma, x : A\ \mathit{true} \vdash_C M : B\ \mathit{true}}{\Gamma \vdash_C \lambda x : A.M : A \supset B\ \mathit{true}} \supset T \qquad \frac{\Gamma \vdash_C M : A\ \mathit{true} \quad \Gamma \vdash_C N : B\ \mathit{true}}{\Gamma \vdash_C \langle M, N \rangle : A \wedge B\ \mathit{true}} \wedge T \qquad \frac{}{\Gamma \vdash_C \star : \top\ \mathit{true}} \top T$$

$$\frac{\Gamma \vdash_C M : A\ \mathit{true}}{\Gamma \vdash_C \mathsf{in}_1\ M : A \vee B\ \mathit{true}} \vee T1 \qquad \frac{\Gamma \vdash_C M : B\ \mathit{true}}{\Gamma \vdash_C \mathsf{in}_2\ M : A \vee B\ \mathit{true}} \vee T2 \qquad \frac{\Gamma \vdash_C K : A\ \mathit{false}}{\Gamma \vdash_C \neg K : \neg A\ \mathit{true}} \neg T$$

$$\frac{\Gamma \vdash_C M : A\ \mathit{true} \quad \Gamma \vdash_C K : B\ \mathit{false}}{\Gamma \vdash_C M; K : A \supset B\ \mathit{false}} \supset F \qquad \frac{\Gamma \vdash_C K : A\ \mathit{false} \quad \Gamma \vdash_C L : B\ \mathit{false}}{\Gamma \vdash_C [K, L] : A \vee B\ \mathit{false}} \vee F \qquad \frac{}{\Gamma \vdash_C \bullet : \bot\ \mathit{false}} \bot F$$

$$\frac{\Gamma \vdash_C K : A\ \mathit{false}}{\Gamma \vdash_C \pi_1 \cdot K : A \wedge B\ \mathit{false}} \wedge F1 \qquad \frac{\Gamma \vdash_C K : B\ \mathit{false}}{\Gamma \vdash_C \pi_2 \cdot K : A \wedge B\ \mathit{false}} \wedge F2 \qquad \frac{\Gamma \vdash_C M : A\ \mathit{true}}{\Gamma \vdash_C \neg M : \neg A\ \mathit{false}} \neg F$$

$$\frac{\Gamma \vdash_C M : A\ \mathit{true} \quad \Gamma \vdash_C K : A\ \mathit{false}}{\Gamma \vdash_C \langle M \rhd K \rangle : \#} \# \qquad \frac{}{\Gamma, x : A\ \mathit{true} \vdash_C x : A\ \mathit{true}} \mathsf{hyp}T \qquad \frac{}{\Gamma, x : A\ \mathit{false} \vdash_C x : A\ \mathit{false}} \mathsf{hyp}F$$

$$\frac{\Gamma, u : A\ \mathit{false} \vdash_C E : \#}{\Gamma \vdash_C u : A\ \mathit{false}.E : A\ \mathit{true}} \mathsf{PBC}T \qquad \frac{\Gamma, u : A\ \mathit{true} \vdash_C E : \#}{\Gamma \vdash_C u : A\ \mathit{true}.E : A\ \mathit{false}} \mathsf{PBC}F$$