Midterm 1 SOLUTION

15-317: Constructive Logic

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Instructions

- This exam is closed book and closed Internet. Handwritten notes are permitted. The last page of the exam recaps some rules you may find useful.
- There are four problems, most with several parts. Not all problems are the same size or difficulty, so it may help to read through the whole exam first. You have 80 minutes to complete the exam.
- When writing proofs, remember to label each inference with the rule used and any variables or parameters discharged (e.g., $\supset I^u$).
- You may find it helpful to construct your proofs on scratch paper (such as the back of a page) before writing it clearly in the space provided.
- Most importantly,

DON'T PANIC

Good luck!

	Problem 1	Problem 2	Problem 3	Problem 4	**Total**
Score					
Max	40	40	40	30	150

1 Natural Deduction and Harmony (40 points)

Consider a new connective ♠ defined by the following introduction and elimination rules.

$$\frac{A\ \mathit{true}}{A\spadesuit B\ \mathit{true}}\ \spadesuit I \qquad\qquad \frac{A\spadesuit B\ \mathit{true} \quad \begin{array}{c}[A\ \mathit{true}]\\ \vdots \\ C\ \mathit{true}\end{array} \quad \begin{array}{c}[B\ \mathit{true}]\\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\ \spadesuit E$$

Task 1 (20 points)**.** Are these rules locally sound? If so, give proof; if not, explain why not.

Solution. Yes, the rules are locally sound.

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array}}{A\spadesuit B\ \mathit{true}}\ \spadesuit I \quad \begin{array}{c}[A\ \mathit{true}]\\ \mathcal{D}_1 \\ C\ \mathit{true}\end{array} \quad \begin{array}{c}[B\ \mathit{true}]\\ \mathcal{D}_2 \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\ \spadesuit E \quad \Longrightarrow \quad \begin{array}{c}[\mathcal{D}]\\ \mathcal{D}_1 \\ C\ \mathit{true}\end{array}$$

■

Task 2 (20 points)**.** Are these rules locally complete? If so, give proof; if not, explain why not.

Solution. No, the rules are not locally complete. If we try to expand $\begin{array}{c}\mathcal{D}\\ A\spadesuit B\end{array}$, we get stuck:

$$\frac{\begin{array}{c}\mathcal{D}\\ A\spadesuit B\ \mathit{true}\end{array} \quad \dfrac{\begin{array}{c}[A\ \mathit{true}]\\ | \\ A\ \mathit{true}\end{array}}{A\spadesuit B\ \mathit{true}}\ \spadesuit I \quad \begin{array}{c}[B\ \mathit{true}]\\ ? \\ A\spadesuit B\ \mathit{true}\end{array}}{A\spadesuit B\ \mathit{true}}\ \spadesuit E$$

■

2 Normal Natural Deduction (40 points)

Recall the $\uparrow\downarrow$ rule for normal natural deduction:

$$\frac{A \downarrow}{A \uparrow} \uparrow\downarrow$$

Suppose we change this rule to make a restriction similar to that we made for init in the sequent calculus—that is, it only applies at *atomic* propositions:

$$\frac{P \downarrow}{P \uparrow} \uparrow\downarrow$$

Claim: the original rule is still admissible. That is, for any proposition A, if $A \downarrow$, then $A \uparrow$. The proof follows by induction on A. (The normal natural deduction rules you'll need for this problem are recapped in Figure 1.)

Task 1 (20 points)**.** State and prove the case for $\wedge$.

Solution. Suppose $A = A_1 \wedge A_2$.

$$\begin{array}{cc} A_1 \downarrow & A_2 \downarrow \\ \vdots & \vdots \end{array}$$

By induction, $A_1 \uparrow$ and $A_2 \uparrow$.

So:

$$\dfrac{\begin{array}{cc} \dfrac{A_1 \wedge A_2 \downarrow}{A_1 \downarrow} \wedge\!\downarrow_1 & \dfrac{A_1 \wedge A_2 \downarrow}{A_2 \downarrow} \wedge\!\downarrow_2 \\ \vdots & \vdots \\ A_1 \uparrow & A_2 \uparrow \end{array}}{A_1 \wedge A_2 \uparrow} \wedge\!\uparrow$$

■

Task 2 (20 points)**.** State and prove the case for ⊃.

Solution. Suppose $A = A_1 \supset A_2$.

$$\begin{array}{cc} A_1 \downarrow & A_2 \downarrow \\ \vdots & \vdots \end{array}$$

By induction, $A_1 \uparrow$ and $A_2 \uparrow$.

So:

$$\dfrac{\dfrac{A_1 \supset A_2 \downarrow \qquad \begin{array}{c}[A_1 \downarrow]^u \\ \vdots \\ A_1 \uparrow\end{array}}{A_2 \downarrow} \supset\downarrow \\ \vdots \\ A_2 \uparrow}{A_1 \supset A_2 \uparrow} \supset\uparrow^u$$

■

3 Nefarious Natural Numbers (40 points)

Consider the following inference rules defining doubling natural numbers as a new class of base propositions:

$$\dfrac{}{\mathsf{double}(0) = 0\ \mathit{true}}\ \mathsf{double}_0 \qquad \dfrac{\mathsf{double}(n) = n'\ \mathit{true}}{\mathsf{double}(s(n)) = s(s(n'))\ \mathit{true}}\ \mathsf{double}_s$$

Task 1 (15 points)**.** Prove in natural deduction that all natural numbers have a double, i.e., $\forall x{:}\mathsf{nat}.\exists y{:}\mathsf{nat}.\mathsf{double}(x) = y$. (The rules for natural number induction are recapped in Figure 2.)

Solution.

$$\dfrac{\dfrac{\begin{array}{c}[x : \mathsf{nat}]\\ \vert \\ x : \mathsf{nat}\end{array} \qquad \dfrac{\dfrac{}{\mathsf{double}(0) = 0\ \mathit{true}}\ \mathsf{double}_0 \quad \dfrac{}{0 : \mathsf{nat}}\ \mathsf{nat}I_0}{\exists y{:}\mathsf{nat}.\mathsf{double}(0) = y\ \mathit{true}}\ \exists I \qquad \begin{array}{c}\mathcal{D}_s\\ \exists y{:}\mathsf{nat}.\mathsf{double}(s(z)) = y\ \mathit{true}\end{array}}{\exists y{:}\mathsf{nat}.\mathsf{double}(x) = y\ \mathit{true}}\ \mathsf{nat}E^{z,u}}{\forall x{:}\mathsf{nat}.\exists y{:}\mathsf{nat}.\mathsf{double}(x) = y\ \mathit{true}}\ \forall I^x$$

Where $\mathcal{D}_s$ =

$$\dfrac{\begin{array}{c}[z : \mathsf{nat}][\exists y{:}\mathsf{nat}.\mathsf{double}(z) = y\ \mathit{true}]^u\\ \vert \\ \exists y{:}\mathsf{nat}.\mathsf{double}(z) = y\ \mathit{true}\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}[\mathsf{double}(z) = y\ \mathit{true}]^v\\ \vert \\ \mathsf{double}(z) = y\ \mathit{true}\end{array}}{\mathsf{double}(s(z)) = s(s(y))\ \mathit{true}}\ \mathsf{double}_s \quad \dfrac{\dfrac{\begin{array}{c}[y : \mathsf{nat}]\\ \vert \\ y : \mathsf{nat}\end{array}}{s(y) : \mathsf{nat}}\ \mathsf{nat}I_s}{s(s(y)) : \mathsf{nat}}\ \mathsf{nat}I_s}{\exists y{:}\mathsf{nat}.\mathsf{double}(s(z)) = y\ \mathit{true}}\ \exists I}{\exists y{:}\mathsf{nat}.\mathsf{double}(s(z)) = y\ \mathit{true}}\ \exists E^{y,v}$$

■

Suppose we call the above proof $\mathcal{D}$. Using it, we construct a purported proof that there is a natural number that doubles *all* natural numbers:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}}{\forall x{:}\mathsf{nat}.\exists z{:}\mathsf{nat}.\mathsf{double}(x) = z\ \mathit{true}} \qquad [y : \mathsf{nat}]}{\exists z{:}\mathsf{nat}.\mathsf{double}(y) = z\ \mathit{true}}\ \forall E \qquad \dfrac{[z : \mathit{nat}] \quad [\mathsf{double}(y) = z\ \mathit{true}]^u}{\mathsf{double}(y) = z\ \mathit{true}}}{\mathsf{double}(y) = z\ \mathit{true}}\ \exists E^{z,u}}{\forall y{:}\mathsf{nat}.\mathsf{double}(y) = z\ \mathit{true}}\ \forall I^y}{\exists x{:}\mathsf{nat}.\forall y{:}\mathsf{nat}.\mathsf{double}(y) = x\ \mathit{true}}\ \exists I$$

Task 2 (15 points)**.** Circle the rule name of each invalid inference. Briefly explain why this proof is wrong.

Solution. The blue-highlighted rules are both applied incorrectly. First, the $\exists I$ is missing a premise — but the reason it's missing that premise is because of the more critical problem in the use of $\exists E$, where the variable z escapes its scope. Remember that the scope of a hypothesis is the subtree of the derivation where it is introduced. ■

Task 3 (10 points)**.** Write out the proof term for the above deduction and point out the problem in the term.

Solution. (It was clarified during the exam that we should use D to stand for the derivation from two tasks previous.)

The proof term is

$\langle z, \Lambda y{:}\mathsf{nat}.\mathsf{let}\ \langle z, u\rangle = D\ y\ \mathsf{in}\ u\rangle$

The problem is that the first occurrence of z is out of scope. ■

4 Unprovability (30 points)

Task 1. Give a proof that $P \supset (P \supset Q)$ is not provable in natural deduction.

Solution. (It was clarified during the exam that P and Q are atomic propositions.)

Suppose $P \supset (P \supset Q)$ *true* is derivable in natural deduction.
Then by the correspondence theorem we proved in class, $\cdot \Longrightarrow P \supset (P \supset Q)$ is provable in the sequent calculus.
Let $\mathcal{D}$ be a derivation of $\cdot \Longrightarrow P \supset (P \supset Q)$.
By inspection of the sequent calculus rules, $\mathcal{D}$ must have the form

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}' \\ P, P \Longrightarrow Q\end{matrix}}{P \Longrightarrow P \supset Q} \supset R}{\cdot \Longrightarrow P \supset (P \supset Q)} \supset R$$

By inspection, no sequent rule can prove $P, P \Longrightarrow Q$. Therefore $\mathcal{D}$ cannot exist. ∎

A Useful Rules

$$\frac{A \uparrow \quad B \uparrow}{A \wedge B \uparrow} \wedge\!\uparrow \qquad \frac{A \wedge B \downarrow}{A \downarrow} \wedge\!\downarrow_1 \qquad \frac{A \wedge B \downarrow}{B \downarrow} \wedge\!\downarrow_2$$

$$\frac{\begin{array}{c}[A \downarrow]^u \\ \vdots \\ B \uparrow\end{array}}{A \supset B \uparrow} \supset\!\uparrow^u \qquad \frac{A \supset B \downarrow \quad A \uparrow}{B \downarrow} \supset\!\downarrow$$

Figure 1: Normal natural deduction rules for ∧ and ⊃.

$$\frac{}{0 : \mathsf{nat}} \mathsf{nat}I_0 \qquad \frac{n : \mathsf{nat}}{\mathsf{s}\ n : \mathsf{nat}} \mathsf{nat}I_{\mathsf{s}} \qquad \frac{n : \mathsf{nat} \quad C(0)\ \mathit{true} \quad \begin{array}{c}[x : \mathit{nat}]\ [C(x)\ \mathit{true}]^u \\ \vdots \\ C(\mathsf{s}\ x)\ \mathit{true}\end{array}}{C(n)\ \mathit{true}} \mathsf{nat}E^{x,u}$$

$$\frac{n : \mathsf{nat}}{n = n\ \mathit{true}} \mathsf{refl}$$

Figure 2: Rules for natural numbers and induction.