15-251: Great Theoretical Ideas In Computer Science

Recitation 12

November 17, 2014

1. **Turing Machine**
   (a) Prove that $\{0^n 1^n 0^n \mid n \in \mathbb{N}\}$ is decidable.
   (b) Prove that $\{M \mid L(M) = 0^n 1^n 0^n\}$ is undecidable.

2. **More Turing Machine**

   Recall that a language L is decidable iff. there exists a TM such that

   - Accept all $w \in L$
   - Reject all $w \notin L$

   Define a language L Turing-Acceptable iff. there exists a TM such that

   - Accept all $w \in L$
   - Rejct or loop forever on all $w \in L$

   (a) Prove or disprove: The complement of a decidable language is decidable.
   (b) Prove or disprove: The complement of a Turing-Acceptable language is Turing-Acceptable.

Definitions and Theorems from the Gödel Lecture

Soundness: A logic is sound if every theorem is true. Intuitively, it means, *if I can prove something, then it must be true.*

Completeness: A logic is complete if for every sentence S, either S or $\neg S$ is a theorem. Intuitively, it means, *I can prove everything that is true.*

Consistency: A logic is consistent if for every sentence S, at least one of S or $\neg S$ is not a theorem. Intuitively, it means, *I cant prove contradictions!*

Gödels Completeness Theorem: There exists an axiomatic system with computable axioms whose theorems are precisely the set of valid sentences in first-order logic.

Gödel's First Incompleteness Theorem: Any mathematical proof system which can define Turing Machines and has computable axioms cannot be both complete and sound.

Gödels Second Incompleteness Theorem: Any mathematical proof system which can define Turing Machines and has computable axioms cannot be both complete and consistent.

Corollary: If ZFC is consistent, then ZFC is incomplete and it cannot prove the statement "ZFC is consistent."

3. Löbs Theorem

Löbs theorem states that if ZFC (or any sufficiently expressive system) can prove its own soundness, then it is inconsistent. This is rather surprising, since it basically means that we cannot prove that our proofs are correct!

To prove Löbs theorem, we will define the following:

- Let S be an arbitrary statement (i.e. $2 + 2 = 4$, $\mathrm{P} = \mathrm{N\ P}$, etc.)
- Let A be the statement "There is a proof of S within ZFC."
- Let T be the statement "$A \rightarrow S$", i.e. ZFC is sound.

To prove Löbs Theorem, consider the axiomatic system Z^*, which is ZFC with the axiom $\neg S$.

For the rest of the proof, assume that there is a ZFC proof of T.

(a) Show that in Z^*, it is possible to prove $\neg A$.

(b) Show that in Z^*, it is possible to prove that ZFC is consistent.

(c) Show that in Z^*, it is possible to prove that Z^* is consistent.

(d) Deduce that there is a ZFC proof of S.

(e) Prove Löbs theorem.