1. **Ramsey**

   Show that every graph with 6 vertices has either a 3-clique or a 3-independent set.

   > **Solution:** Consider a complete graph on six vertices, K_6. The problem statement is equivalent to showing that any edge 2-coloring of K_6 has a triangle of 3 edges of one color or a triangle of the other color. Let the two colors be red and blue. Pick a vertex v in K_6. There are 5 edges incident on this vertex, and by pigeonhole principle at least 3 of these edges are the same color. Without loss of generality, let three edges be blue, which link v to vertices i, j, k. Consider the edges $(i, j), (j, k), (k, i)$. If any of them are blue, then we have a blue triangle. If all of them are red, then we have a red triangle. Done.

2. **Hamilton**

   A mouse eats his way through a 3x3x3 cube of cheese by tunnelling through all of the 27 1x1x1 subcubes. If he starts at one corner and always moves on to an uneaten subcubes. If he starts at one corner and always moves on to an uneaten subcube, can he finish at the center of the cube?

   > **Solution:** The problem statement is equivalent to determining whether there exists a Hamiltonian path on the 3-cube starting from a corner and ending at the center. We can describe the 3-cube as 27 lattice points. Without loss of generality, let the corner in which the mouse starts be the point $(0, 0, 0)$. Note that every edge along a path in the 3-cube will change the parity of the coordinate the mouse is at. A Hamiltonian path that traverses 27 vertices must contain exactly 26 edges. Therefore the origin and terminus of any Hamiltonian path on a 3-cube will be of the same parity. Since the center point is at $(1, 1, 1)$, there is no Hamiltonian path that starts at a corner and ends at the center. Done.

3. **Konig**

   A *line* of a matrix is a row or a column of the matrix. Show that the minimum number of lines containing all of the 1's of a $(0, 1)$-matrix i equal to the maximum number of 1's, no two of which are in the same line.

   > **Solution:** Let the (0, 1)-matrix represent the adjacency of vertices in a bipartite graph. The minimum number of lines containing all of the 1's is equivalent to the minimum number of chosen vertices such that every edge is incident on a chosen vertex, i.e. a minimum vertex cover. The maximum number of 1's such that no two are on the same line is equivalent to the maximum number of chosen edges such that no edges share a vertex, i.e. a maximum matching. By Konig's theorem, in a

bipartite graph the number of edges in a maximum matching is equal to the number of vertices in a minimum covering. Thus the minimum number of lines to cover all 1's and maximum number of line-indpendent 1's are equal.

4. **Hall**

A round-robin tournament among $2n$ teams lasted for $2n-1$ days. Each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Is it possible to choose one winning team from each day without choosing any team more than once?

Solution: Create a bipartite graph in the following manner: In the first partition T, we have a vertex for each team, such that $|T| = 2n$. In the second partition D, we have a vertex for each day, such that $|D| = 2n - 1$. Create an edge between a team vertex and a day vertex if that team won its game on that day. Thus each vertex in D will have degree n. The problem statement is equivalent to showing that there exists a matching of size $2n - 1$, i.e. for every day there is a unique team that won on that day. By Hall's theorem, we know that such a matching exists iff $\forall S \subseteq D, |N(S)| \geq |S|$. Assume for the sake of contradiction that there exists some S for which $|N(S)| < |(S)|$. Then there is at least one team $t \in T \setminus N(S)$. Then t must have lost on every day in S, and since no pair of teams play more than once, there are at least $|S|$ teams that won against t, which implies $|N(S)| \geq |S|$. Done.