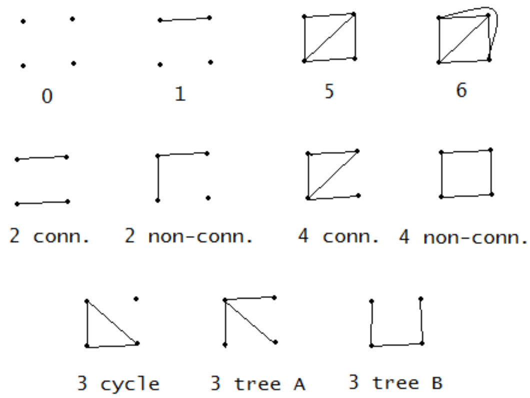
1. **Isomorphisms**

Show that there are eleven nonisomorphic simple graphs on four vertices.

Solution:



2. **Euler's Formula**

A soccer ball is a convex polyhedron whose faces are either hexagons or pentagons. Prove that a soccer ball has exactly 12 pentagonal faces.

Solution: Let the total number of faces on the soccer ball be F, the number of pentagonal faces be f_5, and the number of hexagonal faces be f_6. Consequently, $F = f_5 + f_6$.

First, note that every edge on the soccer ball is adjacent to two faces. Then we have the equation

$$2E = 5f_5 + 6f_6$$

which can be plugged into Euler's formula like so:

$$\begin{aligned} V - E + F - 2 &= V - \frac{5f_5 + 6f_6}{2} + f_5 + f_6 - 2 \\ &= V - \frac{3f_5 + 4f_6}{2} - 2 \end{aligned}$$

Next, note that the sum of the degrees of the angles at a vertex on a convex polyhedon must be less than 360°. Then there must be exactly three faces that meet at each vertex (108° per pentagon, 120° per hexagon). This gives us the equation

$$3V = 5f_5 + 6f_6$$

which can be combined with our previous equation to find the following:

$$\begin{aligned} 3\left(\frac{3f_5 + 4f_6}{2} + 2\right) &= 5f_5 + 6f_6 \\ f_5 &= 12 \end{aligned}$$

3. **Graph Induction**

A tournament of n vertices is a complete directed graph with n vertices. Prove that there always exists a directed path that visits each vertex in the tournament exactly once.

Solution: Induction on the number of vertices in the tournament. If there are two vertices, there is exactly one edge and the directed path is immediately obvious. Consider the case of k vertices. Remove a vertex and suppose any tournament of $k-1$ vertices has a directed Hamiltonian path. If there is an edge pointing from the removed vertex to the tail of the $k-1$ path, we are done. Similarly, if there is an edge pointing from the head of the $k-1$ tail to the removed vertex, we are done. Otherwise, step through the directed path until we find the first vertex i such that there is an edge pointing from the removed vertex to vertex i. Every vertex in the path before i must have an edge pointing to the removed vertex, so we can create a new directed path from the tail to the removed vertex to the ith vertex to the head.