1. **Indicator Random Variables**

100 homeworks are on the table, with two questions to be graded. Andy is in charge of grading question one and Patrick is in charge of grading question two. First, Andy grades some homeworks at random; each homework has probability 0.3 of being graded. Next, Patrick randomly grades half of the homeworks. That is, he grades 50 homeworks. Assume Andy and Patrick make their choices independently.

(a) Let N be the number of homeworks that Andy graded. What is $E[N]$?

Solution:
Let N_i be the indicator RV of "The ith homework being graded by Andy". Therefore,

$$\begin{aligned} E\left[N_i\right] &= 0.3 \\ N &= \sum N_i \\ E\left[N\right] &= E\left[\sum N_i\right] \\ &= \sum E\left[N_i\right] \\ &= 100 \cdot 0.3 \\ &= 30 \end{aligned}$$

(b) Let M be the number of homeworks that Andy graded and Patrick did not grade. What is $E[M]$?

Solution:
The probability of a single homework being graded by Andy but not by Patrick is $0.3 \cdot (1 - 0.5) = 0.15$

Let M_i be the indicator RV of "The ith homework being graded by Andy but not graded by Patrick". Therefore,

$$\begin{aligned} E\left[M_i\right] &= 0.15 \\ N &= \sum M_i \\ E\left[N\right] &= E\left[\sum M_i\right] \\ &= \sum E\left[M_i\right] \\ &= 100 \cdot 0.15 \\ &= 15 \end{aligned}$$

(c) Let a "good pair" be a pair of adjacent homeworks such that both questions are graded. Let P be the number of good pairs. What is $E[P]$?

Solution:
The probability that a pair of homeworks are both graded by Andy is $0.3 \cdot 0.3 = 0.09 = p_a$. The probability that a pair of homeworks are both graded by Patrick is $\frac{50}{100} \cdot \frac{49}{99} = p_b$. Since Andy and Patrick choose independently, a pair being a good pair has probability $p_a \cdot p_b$
There are 99 adjacent pairs. Let P_i be the indicator RV of "The ith pair being a good pair". Therefore,

$$\begin{aligned} E[P_i] &= p_a \cdot p_b \\ N &= \sum P_i \\ E[N] &= E\left[\sum P_i\right] \\ &= \sum E[P_i] \\ &= 99 \cdot p_a \cdot p_b \end{aligned}$$

2. **Conditional Probablity**

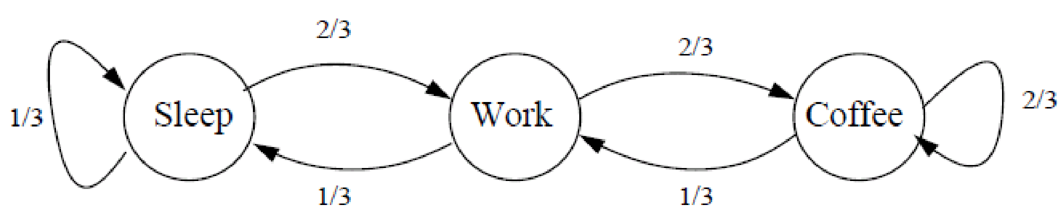
(a) $Pr[A] = 1/2, Pr[B] = 1/3, Pr[A|B] = 1/4$. $Pr[A \cap B] =?$

Solution: $Pr[A \cap B] = Pr[A|B] \cdot Pr[B] = 1/12$

(b) $Pr[A|B] = 1/2, Pr[B|A] = 1/5$, A, B are independent. $Pr[A \cap B] =?$

Solution: Because A, B are independent, $P[A] = P[A|B] = 1/2, P[B] = P[B|A] = 1/5$.
Therefore, $Pr[A \cap B] = P[A] \cdot P[B] = 1/10$.

(c) Peter only sleeps, works, or drinks coffee in his life. Each hour, Peter changes his status according to the following graph. Peter is now at work.



Let T represent the number of hours until he goes to sleep. What is $E[T]$?

Solution: Let S be the number of hours until Peter goes to sleep when he's drinking coffee.
By the graph:

$$\begin{aligned} Pr[T = 1] &= 1/3 && \text{(work to sleep)} \\ Pr[T = 1 + S] &= 2/3 && \text{(work to coffee)} \\ Pr[S = 1 + T] &= 1/3 && \text{(coffee to work)} \\ Pr[S = 1 + S] &= 2/3 && \text{(coffee to coffee)} \end{aligned}$$

Rewrite in expectations:

$$\begin{aligned} E[T] &= \frac{1}{3}E[1] + \frac{2}{3}E[1 + S] \\ &= \frac{1}{3} + \frac{2}{3}\left(1 + E[S]\right) \\ E[S] &= \frac{1}{3}E[1 + T] + \frac{2}{3}E[1 + S] \\ &= \frac{1}{3}\left(1 + E[T]\right) + \frac{2}{3}\left(1 + E[S]\right) \end{aligned}$$

Two equations and two variables, we can solve for $E[T] = 9, E[S] = 12$.