1. **Generating Functions**

(a) Find the generating function for the sequence $1, 1, 1, 1, ....$ in closed form.

> **Solution:** This is just $1+x+x^2+....$ Ignoring convergence, we get this is equal to $\frac{1}{1-x}$.

(b) What's the coefficient of $x^{2005}$ in the generating function $G(x) = \frac{1}{(1+x)^2(1-x)^2}$?

> **Solution:** Notice that our generating function is $\frac{1}{(1-x^2)^2}$, which means that we'll only have even powers of $x$ in our power series. Thus, our answer is 0.
>
> More formally, we can solve for the power series, which is $1+(2x^2-x^4)+(2x^2-x^4)^2+....$ which clearly has no odd terms.

(c) We have 20 bags, each bag containing a 5 dollar coin and a 7 dollar coin. If we can use at most one coin from each bag, in how many ways can we make 17 dollars, assuming that all coins are distinguishable (i.e. the 5 dollar coin from the first bag is considered to be different from that in the second bag, and so on)?

> **Solution:** This is just $(1+x^5+x^7)^{20}$, and we want to find the coefficient of $x^{17}$.
>
> The only way to get 17 is with two $x^5$s and one $x^7$.
>
> There are $\binom{20}{2}$ ways of choosing the $x^5$ terms and 18 ways of choosing the $x^7$ term, giving $190 * 18 = 3420$.

(d) Using generating functions, find $a_n$ in terms of $n$:

$a_0 = 2$ and $a_{n+1} = 3a_n$ for $n \geq 0$.

**Solution:**
We have $G(x) = a_0 + a_1 x + a_2 x^2 + ...$

$3xG(x) = 3a_0 + 3a_1 x + ....$

$(1-3x)G(x) = (1-3x)a_0 + (1-3x)a_1 x + ...$

Since $a_0 = 2$ and $a_{n+1} = 3a_n$, we get $(1-3x)G(x) = 2$, so

$G(x) = \frac{2}{1-3x}$

$= 2[1 + 3x + (3x)^2 + ....]$

2. **Nim with Stones**

We play in a line of squares labeled 0, 1, 2, with several stones are placed in some square such that no two stones are placed in a same square. One move consists of moving one stone to its left onto any empty square and not passing any other stone.

The game ends when a player cannot make any legal moves, since all the stone are jammed at the left-end of the strip.

Find the $P$ and $N$ positions of this game.

**Solution:**
Let the stones on the field be labeled $a_1, a_2, ..., a_n$ starting from the left. Let the number of free spaces to the left of $a_1$ be denoted as $p_1$, spaces between $a_1$ and $a_2$ be $p_2$, and so on until $p_n$.

We now start from $p_n$ and keep piles $p_n$, $p_{n-2}$, and so on, deleting every other $p_i$ value. For example, if our stones were on $2, 5, 10, 12$, our corresponding $p_i$ values would be $1, 2, 4, 1$, and the $p_i$ values we keep end up being $2, 1$. We now claim that the remaining $p_j$ values correspond to a variation of nim.

The $p_j$ values that remain correspond to a pile of $p_j$ stones, where moving $a_j$ $k$ spaces to the left is the same as removing $k$ stones from the pile, and moving $a_{j-1}$ $l$ spaces to the left correspond to adding $l$ stones to the pile. We note that we can only add stones to the pile a finite number of times, since the number of times we can move $a_{j-1}$ to the left is finite. Our goal is to shift all of the stones to the left, which is effectively trying to make the number of stones in every pile zero. This game corresponds to the winning positions of nim, where the nim-sum of all the $p_j$'s that we keep is non-zero.

We note that adding stones to the pile does not affect the winning positions of nim, as if the other person adds $b$ stones to the pile, we can simply remove $b$ stones from the same pile, effectively cancelling out the two moves, thus staying at a winning position. Since the number of times he/she can add stones is finite, this will eventually turn exactly into the game of nim.

Thus, since this corresponds to the game of nim, the winning positions of this game is when the nim sum of the piles is non-zero, where the piles are the kept $p_j$ values that we defined earlier in the problem.

Sidenote: Note that we keep the right most $p_i$ pile because if we start from the left and remove every other pile, the right most stone might not correspond to a pile.

3. **Nim with Arrows**

Given a horizontal line of $N$ arrows with some arrows pointing in the right direction, and some pointing in the wrong direction. Each turn, a player have to flip one arrow from wrong to right, and in the same time (if he/she wants), flip one more arrow to the left of it.

The player who flips all the arrows to the right direction wins.

Find the $P$ and $N$ positions of this game.

**Solution:**

We claim that this game is equivalent to nim, where each arrow in position $i$ which is facing in the wrong direction is equivalent to a pile of $i$ stones, and an arrow which is facing in the w direction in position $j$ is equivalent to a pile of 0 stones. Making all the arrow face the right direction is equivalent to making all piles of stones equal to zero.

We now consider all possible moves in this game and will find the corresponding move in nim:

There are three possible moves in nim:

If we turn an arrow in position $j$ from the wrong position to the right position , it is equivalent to removing the whole pile of $j$ stones.

If we turn an arrow in position $j$ from the wrong position to the right position, and an arrow in position $k$ to the left of it from the right position to the wrong position, it is equivalent to removing enough stones from the pile of $j$ stones until only $k$ stones remain in that pile.

If we turn an arrow in position $j$ from the wrong position to the right position, and an arrow in position $k$ to the left of it from the wrong position to the right position, it is equivalent to removing enough stones from the pile of $j$ stones until only $k$ stones remain in that pile, and keeping the original pile of k stones from track $k$ the same.

We note that having two piles of stones with the same number $k$ is the same as having neither pile, since whatever move your opponent does to one pile, we can replicate it to the other pile.

Thus, we can delete both piles $k$ from the third move, while preserving the winning positions of the game of nim. The resulting piles correspond to the direction of the arrows, and thus the winning position/strategy of this game corresponds to the winning position/strategy of nim.

Thus, the player should go first when the nim-sum of the arrows in the positions that are facing wrong direction is non-zero, as it correlates to the winning position in the game of nim.