1. **Axiomatic Systems**

You're a military commander and your intelligence staff has intercepted some enemy communications. They've determined that all of the messages the enemy sends are strings in the set $\{+,-\}^*$. They've also gleaned that every message s that the enemy sends satisfies the following properties:

1. Every prefix of s has at least as many $+$ symbols as $-$ symbols.
2. Every suffix of s has at least as many $-$ symbols as $+$ symbols.

As part of your attempt to decipher your enemy's code, you decide it is worthwhile to try sending messages in their code to them. You therefore need to construct messages s, but you have to make sure that every message you can construct looks like an enemy message (satisfies the properties above). Moreover, you feel it is necessary to be able to construct every such message that the enemy could possible send.

Your task is to give a logical system, or a set of axioms and deduction rules, for messages with the two properties above. Prove that every message that your logical system can derive satisfies the two rules, and that every message that satisfies the two rules can be derived by your logical system. In other words, give a logic for this concept and prove its soundness and completeness.

Solution:

2. **Propositional Logic**

(a) Let R, Q be propositional formulae. Describe a formula P that is a tautology iff R and Q are equisatisfiable.

Solution:

(b) Let $P_1, \ldots, P_k$ be a collection of propositional formulae. We say that this collection is consistent if there is a truth assignment which makes all P_i true. Write a formula S that is a tautology iff this collection is not consistent.

Solution:

3. **First-order Logic**

Express each of the following predicates and propositions in first-order logic, where the universe is taken to be $\mathbb{N}$, the relations are $+(x, y, z)$, $*(x, y, z)$, and $=(x, y)$, where $+(x, y, z)$ is true iff $x + y = z$, and similarly for $*$, and there are no constants.

(a) n is the sum of two squares.

Solution:

(b) $n = 1$

Solution:

(c) m divides n ($m \mid n$).

Solution:

(d) n is prime.

Solution:

(e) n is a power of 2.

Solution: