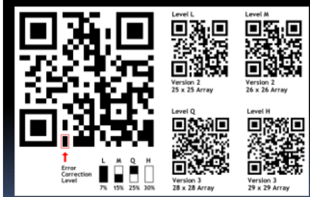
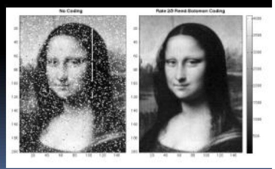
15-251: Great Theoretical Ideas in Computer Science

Lecture 17

October 23, 2014

Error Correction





Polynomial Interpolation

Theorem:

Let arbitrary pairs (a_1,b_1), (a_2,b_2), …, (a_{d+1},b_{d+1}) from a field F be given (with all a_i's distinct). Then there always exists a polynomial P(x) of degree $\leq$ d with $P(a_i) = b_i$ for all i.

Lagrange Interpolation

a_1	b_1
a_2	b_2
a_3	b_3
…	…
a_d	b_d
a_{d+1}	b_{d+1}

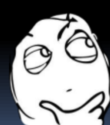
Want P(x)

(with degree $\leq$ d)

such that $P(a_i) = b_i$ $\forall i$.

Lagrange Interpolation

a_1	1
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	0



Can we do this special case?

Lagrange Interpolation

a_1	1
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	0

Numerator is a deg. d polynomial

Denominator is a nonzero field element

Idea #2:

$$S_1(x) = \frac{(x-a_2)(x-a_3)\cdots(x-a_{d+1})}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_{d+1})}$$

Call this the selector polynomial for a_1.

---

a_1	0
a_2	1
a_3	0
…	…
a_d	0
a_{d+1}	0

What about above data?

$$S_2(x) = \frac{(x-a_1)(x-a_3)\cdots(x-a_{d+1})}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_{d+1})}$$

a_1	0
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	1

And for this data,

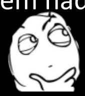
$$S_{d+1}(x) = \frac{(x-a_1)(x-a_2)\cdots(x-a_d)}{(a_{d+1}-a_1)(a_{d+1}-a_2)\cdots(a_{d+1}-a_d)}$$

Polynomial Interpolation

a_1	b_1
a_2	b_2
a_3	b_3
…	…
a_d	b_d
a_{d+1}	b_{d+1}

$$\boxed{P(x) = b_1 \cdot S_1(x) + b_2 \cdot S_2(x) + \cdots + b_{d+1} \cdot S_{d+1}(x)}$$

The Chinese Remainder Theorem had a very similar proof



Not a coincidence:
algebraically, integers & polynomials share many common properties

Lagrange interpolation is the *exact analog* of Chinese Remainder Theorem for polynomials.

Chinese Remainder Theorem: Suppose $n_1, n_2, \ldots, n_k$ are pairwise coprime. Then, for all integers $a_1, a_2, \ldots, a_k$, there exists an integer x solving the below system of simultaneous congruences

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_k \pmod{n_k}. \end{aligned}$$

Further, all solutions x are congruent modulo $N = \prod_{i=1}^{k} n_i$.

Let $m_i = N/n_i$

i'th "selector" number: $T_i = (m_i^{-1} \bmod n_i)\, m_i$

$$x = a_1 T_1 + a_2 T_2 + \ldots + a_k T_k$$

Recall: Interpolation

Let pairs $(a_1,b_1), (a_2,b_2), \ldots, (a_{d+1},b_{d+1})$ from a field F be given (with all a_i's distinct).

Theorem:

There is a unique degree d polynomial $P(x)$ satisfying $P(a_i) = b_i$ for all i = 1…d+1.

A linear algebra view

Let $p(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_d x^d$
Need to find the coefficient vector $(p_0, p_1, \ldots, p_d)$

$p(a) = p_0 + p_1 a + \ldots + p_d a^d$
$= 1 \cdot p_0 + a \cdot p_1 + a^2 \cdot p_2 + \ldots + a^d \cdot p_d$

Thus we need to solve:

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^d \\ 1 & a_2 & a_2^2 & \cdots & a_2^d \\ & & \vdots & & \\ 1 & a_{d+1} & a_{d+1}^2 & \cdots & a_{d+1}^d \end{pmatrix} \cdot \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_d \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{d+1} \end{pmatrix}$$

Lagrange interpolation

The $(d+1)\times(d+1)$ Vandermonde matrix

$$M = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^d \\ 1 & a_2 & a_2^2 & \cdots & a_2^d \\ & & \vdots & & \\ 1 & a_{d+1} & a_{d+1}^2 & \cdots & a_{d+1}^d \end{pmatrix}$$

is **invertible**.

- The determinant of M is nonzero when a_i's are distinct.

Thus can recover coefficient vector **as** $\vec{p} = M^{-1}\vec{b}$

The columns of M^{-1} are given by the coefficients of the various "selector" polynomials we constructed in Lagrange interpolation.

Representing Polynomials

Let P(x)∈F[x] be a degree-d polynomial.

Representing P(x) using d+1 field elements:

1. List the d+1 coefficients.
2. Give P's value at d+1 different elements.

Rep 1 to Rep 2: Evaluate at d+1 elements

Rep 2 to Rep 1: Lagrange Interpolation

Application of Fields/Polynomials (and linear algebra): Error-correcting codes

Sending messages on a noisy channel

Alice → Bob: " bit.ly/vrxUBN "

The channel may corrupt up to k symbols.

How can Alice still get the message across?

Sending messages on a noisy channel

Let's say messages are sequences from $\mathbb{F}_{257}$

vrxUBN ↔ 118 114 120 85 66 78

↓ noisy channel

vrxUBN ↔ 118 114 104 85 35 78

The channel may corrupt up to k symbols.

How can Alice still get the message across?

Sending messages on a noisy channel

Let's say messages are sequences from $\mathbb{F}_{257}$

vrxUBN ↔ 118 114 120 85 66 78

↓ noisy channel

vrxUBN ↔ 118 114 104 85 35 78

How to correct the errors?

How to even detect that there *are* errors?

Simpler case: “Erasures”

118 114 120 85 66 78

$\downarrow$ erasure channel

118 114 ?? 85 ?? 78

What can you do to handle up to k erasures?

Repetition code

Have Alice repeat each symbol k+1 times.

118 114 120 85 66 78

becomes

118 118 118 114 114 114 120 120 120 85 85 85 66 66 66 78 78 78

$\downarrow$ erasure channel

118 118 118 ?? ?? 114 120 120 120 85 85 85 66 66 66 78 78 78

If at most k erasures, Bob can figure out each symbol.

Repetition code – noisy channel

Have Alice repeat each symbol 2k+1 times.

118 114 120 85 66 78

becomes

118 118 118 114 114 114 120 120 120 85 85 85 66 66 66 78 78 78

$\downarrow$ noisy channel

118 118 118 114 223 114 120 120 120 85 85 85 66 66 66 78 78 78

At most k corruptions: Bob can take majority of each block.

This is pretty wasteful!

To send message of d+1 symbols and guard against k erasures, we had to send (d+1)(k+1) total symbols.

Can we do better?

This is pretty wasteful!

To send message of d+1 symbols and guard against k erasures, we had to send (d+1)(k+1) total symbols.

To send even 1 message symbol with k erasures, ***need*** to send k+1 total symbols.

Maybe for d+1 message symbols with k erasures, d+k+1 total symbols can suffice??

Enter polynomials

Say Alice's message is d+1 elements from $\mathbb{F}_{257}$

118 114 120 85 66 78

Alice thinks of it as the coefficients of a degree-d polynomial $P(x) \in \mathbb{F}_{257}[x]$

$$P(x) = 118x^5 + 114x^4 + 120x^3 + 85x^2 + 66x + 78$$

Now trying to send the degree-d polynomial P(x).

Send it in the Values Representation!

$P(x) = 118x^5 + 114x^4 + 120x^3 + 85x^2 + 66x + 78$

Alice sends P(x)'s values on d+k+1 inputs:
P(1), P(2), P(3), …, P(d+k+1)

This is called the **Reed–Solomon encoding**.

Send it in the Values Representation!

$P(x) = 118x^5 + 114x^4 + 120x^3 + 85x^2 + 66x + 78$

Alice sends P(x)'s values on d+k+1 inputs:
P(1), P(2), P(3), …, P(d+k+1)

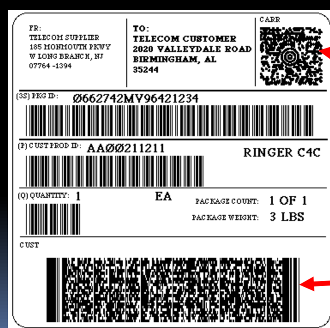
If there are at most k erasures, then
Bob still knows P's value on d+1 points.

Bob recovers P(x) using Lagrange Interpolation!

Example

Reed–Solomon codes are used a lot in practice!

Another everyday use: CD/DVDs, hard discs, satellite communication, …



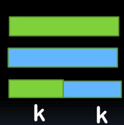
Maxicodes
= "UPS codes"
= another 2-d
Reed-Solomon codes

PDF417 codes
= 2-d Reed-Solomon
codes

What aboout corruptions/errors

To send message of d+1 symbols
and enable correction from up to k errors,
repetition code has to send (d+1)(2k+1) total symbols.

To even communicate 1 symbol while
enabling recovery from k errors,
need to send 2k+1 total symbols.



Maybe for d+1 message symbols with k errors,
d+2k+1 total symbols can suffice??

Want to send a polynomial of degree-d
subject to at most k corruptions.

First simpler problem: Error detection

Suppose we try the same idea

- Evaluate P(X) at d+1+k points
- Send P(0), P(1), P(2), …, P(d+k)

At least d+1 of these values will be unchanged

Example

$P(X) = 2X^2 + 1$, and $k = 1$.

So I sent $P(0)=1$, $P(1)=3$, $P(2)=9$, $P(3)=19$

Corrupted email says (1, 4, 9, 19)

Choosing (1, 4, 9) will give us $Q(X) = X^2 + 2X + 1$

We can now *detect* (up to k) errors

Evaluate P(X) at d+1+k points

Send P(0), P(1), P(2), …, P(d+k)

At least d+1 of these values will be correct

Say P(0), P'(1) , P(2), P(3), P'(4), …, P(d+k)

Using these d+1 correct values will give P(X)

Using any of the incorrect values will give something else

Quick way of detecting errors

Interpolate first d+1 points to get Q(X)

Check that all other received values are consistent with this polynomial Q(X)

If all values consistent, no errors.

In that case, we know Q(X) = P(X)

else there were errors…

How good is our encoding?

Naïve Repetition:
To send d+1 numbers with error detection, sent (d+1)(k+1) numbers

Polynomial Coding:
To send d+1 numbers with error detection, sent (d+k+1) numbers

How about error correction?

requires more work

To send d+1 numbers in such a way that we can correct up to k errors, need to send d+1+2k numbers.

Similar encoding scheme

Evaluate degree-d P(x) at d+1+2k points

Send P(0), P(1), P(2), …, P(d+2k)

At least d+1+k of these values will be correct

Say P(0), P(1) , P(2), P(3), P(4), …, P(d+2k)

Trouble: How do we know which are correct?

Theorem: **P(X) is the unique degree-d polynomial that agrees with the received data on at least d+1+k points**

Clearly, the original polynomial P(X) agrees with data on d+1+k points (since at most k errors, out of total d+1+2k points)

And if a different degree-d polynomial R(X) did so, R(X) and P(X) would have to agree with each other on d+1 points, and hence be the same.

So any such R(X) = P(X)

Theorem: **P(X) is the unique degree-d polynomial that agrees with the received data on at least d+1+k points**

Brute-force Algorithm to find P(X):

Interpolate each subset of (d+1) points

Check if the resulting polynomial agrees with received data on d+1+k pts

Takes too much time…

A fast (cubic runtime) algorithm to decode was given by [Peterson, 1960]

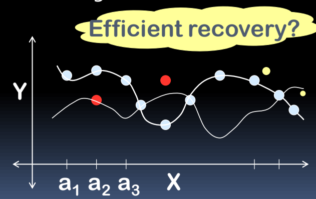
Later improvements by Berlekamp and Massey gave practical algorithms

We will now describe the Welch-Berlekamp algorithm to recover the original polynomial when there are k errors

Aside: Recent research (incl. some of my own) has given algorithms to correct even more than k errors (in a meaningful model)

Reed-Solomon codes

- Message = $(m_0, m_1, \ldots, m_d) \in F^{d+1}$
- $(F = Z_p)$ $\quad$ **$(P(a_1), P(a_2) \ldots P(a_n))$**
- Polynomial curve $Y = P(X) = m_0 + m_1X + \ldots + m_dX^d$
- Encoding = eval. at $n = d+2k+1$ *distinct* $a_i \in F$

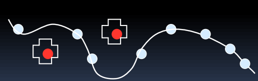


Two curves differ in > 2k positions

Message uniquely identifiable for up to k errors

Error-correction approach

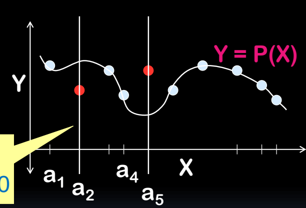
- Given $n = d+2k+1$ points (a_i, y_i) where received value $y_i \neq P(a_i)$ for at most k points.
- If we locate positions of errors, problem then easy by interpolation on correct data



Locating the errors

Noisy points mess up curve-fitting/interpolation

Interpolate more general curve through **all** n points



Here's such a curve.
$(Y-P(X))\,(X-a_2)\,(X-a_5) = 0$

Here $E(X) := (X-a_2)\,(X-a_5)$ is the *error locator polynomial*, with roots at all error locations

Of course, we *don't know* E(X) (that's the whole problem!)

Developing the algorithm

Let Err be subset of k erroneous locations, and define error locator polynomial $E(X) = \prod_{i \in Err} (X - a_i)$

- degree(E) = k

We have $E(a_i)\, y_i = E(a_i)\, P(a_i)$ for i=1,2,…,n

Let $N(X) = E(X)\, P(X)$; degree(N) = d + k.

So $E(a_i)\, y_i - N(a_i) = 0$ for all points (a_i, y_i)

Can we use above to find polynomials E(X) and N(X) (and hence also P(X) = N(X)/E(X))?

Finding N and E

$E(a_i)\, y_i - N(a_i) = 0$ for all points (a_i, y_i)

- $E(X) = X^k + b_{k-1}X^{k-1} + \ldots + b_0 =$
- $N(X) = c_{d+k}\, X^{d+k} + \ldots + c_1 X + c_0$

Finding E(X) and N(X) is same as finding the unknowns $b_0, b_1, \ldots, b_{k-1}, c_0, \ldots, c_{d+k}$

- There are k +(d+k+1) = d+2k+1 = n unknowns
- Also n ***linear*** equations $E(a_i)\, y_i - N(a_i) = 0$ in these n unknowns (why are they linear?)

So we can find E(X) and N(X) by solving this linear system and then output N(X)/E(X)

Spurious solutions?

We know coefficients of E(X) and N(X)=E(X)P(X) are a solution, but what if there are other solutions?

Lemma: If $E_1(X)$ and $N_1(X)$ are a different solution, to $E_1(a_i)\, y_i - N_1(a_i) = 0$ with $\deg(E_1) \le k$, $\deg(N_1) \le d+k$, then $N_1(X)/E_1(X) = P(X)$

Proof: Define $R(X)= E_1(X)P(X)-N_1(X)$

degree(R) ≤ d+k

When $P(a_i)= y_i$,

$R(a_i) = E_1(a_i)P(a_i)-N_1(a_i) = E_1(a_i)y_i - N_1(a_i) = 0$

So R(X) has at least d+k+1 roots. $\Rightarrow R(X) = 0$ □

Thus every solution (E_1,N_1) to the linear system yields the same P(X) as the ratio!

Sending messages on a noisy channel

Alice

Um, what if d+2k+1 > 257?

Message: d+1 symbols from $\mathbb{F}_{257}$

Reed–Solomon: To guard against k corruptions, treat message as coeffs of poly P, send P(1), P(2), …, P(d+2k+1)

Sending messages on a noisy channel

Alice

Um, what if d+2k+1 > 257?

What if the noisy channel corrupts **bits**, not bytes?

(Can we have fewer redundant **bits**?)

$\mathbb{F}_{257}$

…ainst k corruptions,

…sage as coeffs of poly P,

…send P(1), P(2), …, P(d+2k+1)

Sending messages on a noisy channel

Alice wants to send an n-bit message to Bob.

The channel may flip up to k bits.

How can Alice get the message across?

Sending messages on a noisy channel

Alice wants to send an (n−1)-bit message to Bob.

The channel may flip up to 1 bit.

How can Alice get the message across?

Q1: How can Bob detect if there's been a bit-flip?

Parity-check solution

Alice tacks on a bit, equal to the parity of the message's n−1 bits.

Alice's n-bit 'encoding' always has an even number of 1's.

Bob can detect if the channel flips a bit: if he receives a string with an odd # of 1's.

1-bit error-detection for 2^{n-1} messages by sending n bits: optimal! (simple exercise)

Linear Algebra perspective

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$$

G: an n×(n−1) 'generator' matrix

Alice's message $x \in \mathbb{F}_2^{n-1}$

Bob receives

Linear Algebra perspective

Let C be the set of strings Alice may transmit.

C is the span of the columns of G.

C is a subgroup of $\mathbb{F}_2^n$
[In linear algebra terms, an (n−1)-dimensional subspace of the vector space $\mathbb{F}_2^n$]

Linear Algebra perspective

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} \stackrel{?}{=} 0$$

H: a 1×n 'parity check' matrix

Bob receives

Bob checks this to detect if no errors

Solves 1-bit error detection, but not correction

If Bob sees z = (1, 0, 0, 0, 0, 0, 0),

did Alice send y = (0, 0, 0, 0, 0, 0, 0),
or y = (1, 1, 0, 0, 0, 0, 0),
or y = (1, 0, 1, 0, 0, 0, 0),
or… ?

The Hamming(7,4) Code

Alice communicates 4-bit messages (16 possible messages) by transmitting 7 bits.

$$G' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Alice encodes $x \in \mathbb{F}_2^4$ by G' x,

which looks like x followed by 3 extra bits.

The Hamming(7,4) Code

Alice sends 4-bit messages using 7 bits.

Any 'codeword' $y = Gx$ satisfies some 'parity checks':

$$y_1 = y_3 + y_5 + y_7$$
$$y_2 = y_3 + y_6 + y_7$$
$$y_4 = y_5 + y_6 + y_7$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

I.e., $Hy = 0$

Let's permute the output 7 bits (rows of G')

$$G = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Hamming(7,4) Code

Alice communicates 4-bit messages using 7 bits.

Columns are 1…7 in binary!

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$Hy = 0$, because $HG = 0$.

The Hamming(7,4) Code

On receiving $z \in \mathbb{F}_2^7$, Bob computes Hz.

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

If no errors, $z = Gx$, so $Hz = HGx = 0$.

If jth coordinate corrupted, $z = Gx+e_j$.

Then $Hz = H(Gx+e_j) = HGx + He_j$

$= He_j =$ (j'th column of H) = binary rep. of j

Bob knows where the error is, can recover msg!

Sending longer messages: General Hamming Code

By sending n = 7 bits, Alice can communicate one of 16 messages, guarding against 1 error.

This scheme generalizes: Let $n = 2^r-1$, take H to be the $r \times (2^r-1)$ matrix whose columns are the numbers $1 \ldots 2^r-1$ in binary.

There are $2^{n-r} = 2^n/(n+1)$ solutions $z \in \{0,1\}^n$ to the check equations $Hz = 0$.

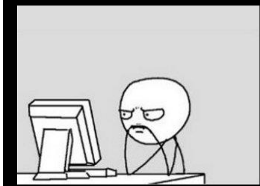
- These are *codewords* of the Hamming code of length n

Summary: Hamming code

To ***detect*** 1 bit error in n transmitted bits:
- one parity check bit suffices,
- so can communicate 2^{n-1} messages by sending n bits.

To ***correct*** 1 bit error in n transmitted bits:
- for $n = 2^r - 1$, r check bits suffice
- so can communicate $2^n/(n+1)$ messages by sending n bits

Fact (left as exercise): This is optimal!



Study Guide

Polynomials:

Lagrange Interpolation

Parallel with Chinese Remainderin

Reed-Solomon codes:

Erasure correction via interpolation

Error correction

Hamming codes:

Correcting 1 bit error