Great Theoretical Ideas In Computer Science

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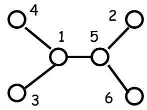
Graphs - II

Cayley's Formula

The number of labeled trees on n nodes is n^{n-2}

Put another way, it counts the number of spanning trees of a complete graph K_n.



P = 5, 1, 1, 5

We proved it by finding a bijection between the set of Prüfer sequences and the set of labeled trees.

Planar Graphs

Theorem: In any connected planar graph with V vertices, E edges and F faces, then

$$V - E + F = 2$$

Theorem: In any connected planar graph with at least 3 vertices:

$$E \leq 3V - 6$$

Lemma: In any connected planar graph with at least 3 vertices:

$$3F \leq 2E$$

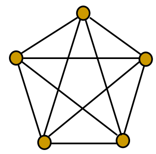
Is K_5 planar?

K_5 has 5 vertices and 10 edges, thus

$$E = 10 \leq 3 \times 5 - 6 = 9$$

which is false, therefore

K_5 is not planar.

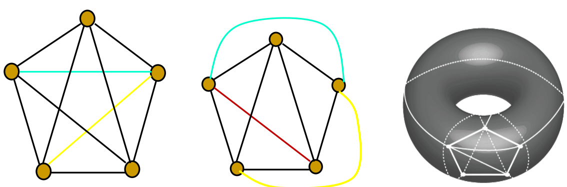


Theorem: In any connected planar graph with V vertices, E edges and F faces, then

$$V - E + F = 2$$

K_5 can be embedded on the torus

Embedding a graph onto a surface means drawing the graph on the surface such that no edges cross.



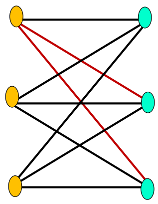
Always there is a surface so any graph can be embedded to.

Outline

- Bipartite Graphs
- Kuratowski-Wagner Theorem
- Graph Coloring
- Bipartite Matching

Bipartite Graphs

A graph is bipartite if the vertices can be partitioned into two sets V_1 and V_2 such that all edges go only between V_1 and V_2 (no edges go from V_1 to V_1 or from V_2 to V_2)

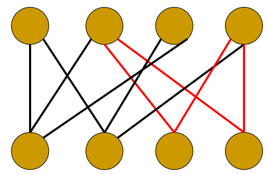


The complete bipartite graphs $K_{m,n}$ have the property that two vertices are adjacent if and only if they do not belong together in the bipartition subsets.

Bipartite Graphs

Theorem. A graph is bipartite iff it does not have an odd length cycle.

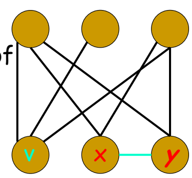
Proof. $\Longrightarrow$)
If it's bipartite and has a cycle, its length must be even.



Bipartite Graphs

Theorem. A graph is bipartite iff it does not have an odd length cycle.

$\Longleftarrow$) (by construction)
Fix a vertex v. Define two sets of vertices
A ={w $\in$V | even length shortest edge path from v to w}
B ={w $\in$V | odd length shortest edge path from v to w}



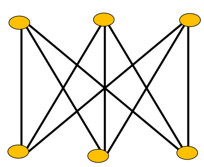
If x and y from A, they cannot be adjacent. By contradiction. There will be an odd length cycle.
The same argument for B. These sets provide a bipartition

Is a tree always a bipartite graph?

Is $K_{3,3}$ planar?

In any connected planar graph with at least 3 vertices:

$E \leq 3V - 6$



Utility Graph

$K_{3,3}$ has 6 vertices and 9 edges, thus

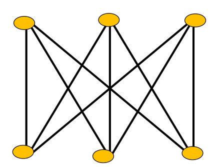
$E = 9 \leq 3x6 - 6 = 12$

Not conclusive!

Is $K_{3,3}$ planar?

$\sum$(edge, face) $\leq$ 2 E, since each edge is associated with at most 2 faces.

$\sum$(edge, face) $\geq$ 4 F , since graph contains no simple triangle regions of 3 edges.



It follows, that
$4F \leq 2E$
and for $K_{3,3}$ we have
$4F \leq 18$
$F \leq 4.5$

From Euler's theorem: $V - E + F = 2$
$F = 2 + 9 - 6 = 5$. Contradiction!

Planar Bipartite Graphs

The previous example established two simple criteria for testing whether a given planar graph is bipartite.

Theorem. In any bipartite planar graph with at least 3 vertices:

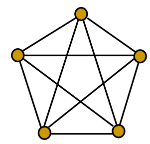
$$E \leq 2V - 4$$

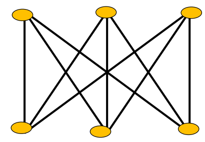
Lemma: In any bipartite planar graph with at least 3 vertices:

$$4F \leq 2E$$

Kuratowski Theorem (1930)

Theorem. A graph is planar if and only if it contains no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$.

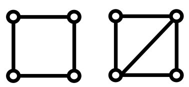




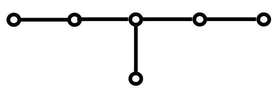
For any graph on V vertices there are efficient algorithms for checking if the graph is planar. The best one runs in linear time O(V)

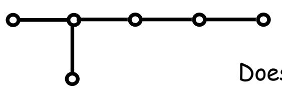
Graph Isomorphism

Definition. Two simple graphs G and H are isomorphic $G \cong H$ if there is a vertex bijection $V_H \to V_G$ that preserves adjacency and non-adjacency structures.



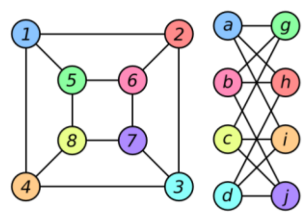
Does not preserve adjacency





Does not preserve adjacency

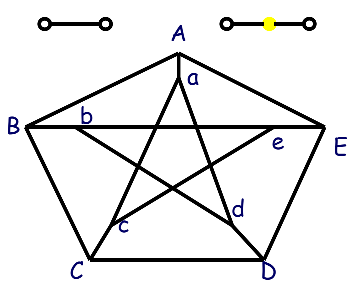
Graph Isomorphism



The graph isomorphism problem has no known polynomial time algorithm which works for an arbitrary graph.

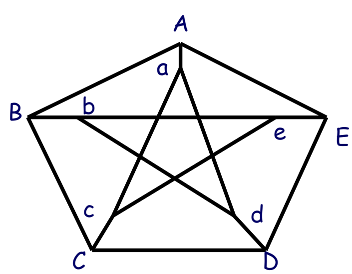
Subdivision

Definition. Subdividing an edge means inserting a new vertex (of degree two) into this edge.



The Petersen graph 10 vertices 15 edges

Theorem. A graph is planar if and only if it contains no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$.



Petersen graph

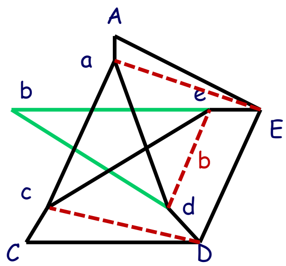
Remove B to get a subgraph

Theorem. A graph is planar if and only if it contains no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$.

A is subdividing (a,E)

b is subdividing (d,e)

C is subdividing (c,D)



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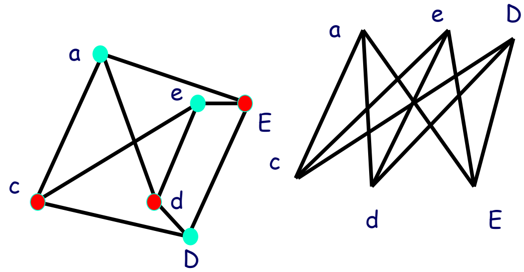


Subdivision and Contraction

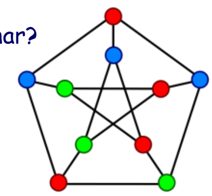
Definition. Subdividing an edge means inserting a new vertex (of degree two) into this edge.

Definition. An edge contraction is an operation which removes an edge from a graph while simultaneously merging the two vertices it used to connect.

Wagner Theorem

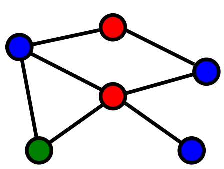
Theorem. Graph G is planar if and only if it contains no subgraph that can be contracted to K_5 or $K_{3,3}$.

Is the Petersen graph planar?



Coloring Graphs

A coloring of a graph is an assignment of a color to each vertex such that no neighboring vertices have the same color



Graph Coloring

Theorem : Every simple planar graph has a vertex of degree < 6.

Proof.

$\sum \deg(v_k) = 2\,E \le 2\,(3\,V - 6)$

Average degree:

$1/V \sum \deg(v_k) \le 6 - 12/V < 6$

Thus, there exists a vertex of degree < 6.
This technique is called the probabilistic method.

Coloring Planar Graphs

Theorem: Any simple **planar** graph can be colored with 6 colors.

Proof. (by induction on the number of vertices).

If G has six or less vertices, then the result is obvious. Suppose that all such graphs with V-1 vertices are 6-colorable

Remove a vertex of degree less than 6, use IH.

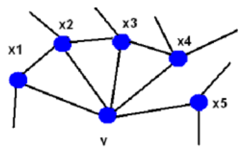
Put it back, since it has at most 5 adjacent vertices, we have enough colors. QED

Coloring Planar Graphs

Theorem: Any simple **planar** graph can be colored with less than or equal to 5 colors.

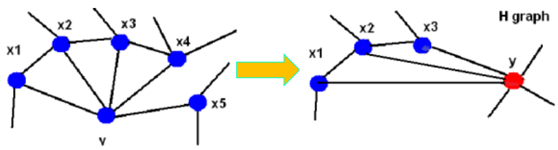
Proof. (repeat the 6-colors proof)

Pick a vertex v of degree 5. Label the vertices adjacent to v as x_1, x_2, x_3, x_4 and x_5. Assume that x_4 and x_5 are not adjacent to each other.



Why we can assume this?

If all x1,…,x5 adjacent, we get K_5.



Contract edges (v, x_4), (v, x_5). Vertices v, x_4, x_5 will be replaced by y, so neighbors of v, x_4, x_5 will be neighbors of y.

We obtain a new graph H with two less vertices. By IH the graph H can be colored with 5 colors.

Next, we assign y-color to x_4 and x_5

We give v a color different from all colors used on the four vertices x_1, x_2, x_3 and y. QED

4 Color Theorem (1976)

Theorem: Any simple **planar** graph can be colored with less than or equal to 4 colors.

It was proven in 1976 by K. Appel and W. Haken. They used a special-purpose computer program.

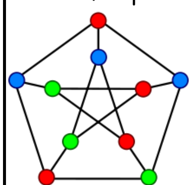
Since that time computer scientists have been working on developing a formal program proof of correctness. The idea is to write code that describes not only what the machine should do, but also why it should be doing it.

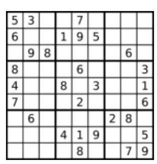
In 2005 such a proof has been developed by Gonthier, using the Coq proof system.

Graph Coloring

Graph coloring is computationally hard. It is NP-complete to decide if a given graph admits k>2 colorings.

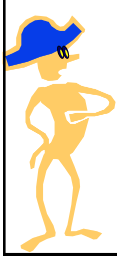
Important case: 3-coloring., also NP-complete, even for planar graphs.





Bipartite Matching

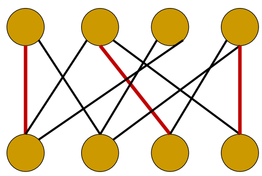
A graph is bipartite if the vertices can be partitioned into two disjoint (also called independent) sets V_1 and V_2 such that all edges go only between V_1 and V_2 (no edges go from V_1 to V_1 or from V_2 to V_2)



Personnel Problem. You are the boss of a company. The company has M workers and N jobs. Each worker is qualified to do some jobs, but not others. How will you assign jobs to each worker?

Bipartite Matching

Definition. A subset of edges is a matching if no two edges have a common vertex (mutually disjoint).

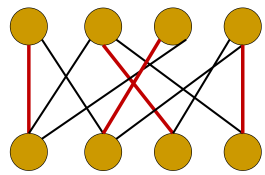


Definition. A maximum matching is a matching with the largest possible number of edges

Bipartite Matching

Definition. A perfect matching is a matching in which each node has exactly one edge incident on it.

A perfect matching is like a bijection, which requires that $|V_1| = |V_2|$ and in which case its inverse is also a bijection.



Hall's (marriage) Theorem

Theorem. (without proof)
Let G be bipartite with V_1 and V_2.
For any set $S \subset V_1$, let $N(S)$ denote the set of vertices adjacent to vertices in S.

Then, G has a perfect matching if and only if

$$|S| \leq |N(S)|$$

for every $S \subset V_1$.

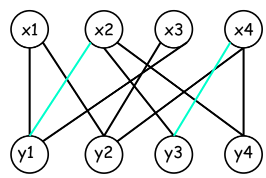
Maximum Matching

Consider a matching M = {(y1,x2),(y3,x4)}

Alternating path has edges alternating between M and E/M. Path x1, y1, x2, y3, x4 is alternating.

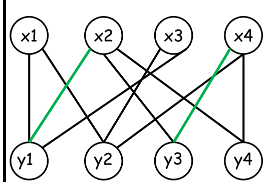
An alternating path is augmenting if both of its endpoints are free vertices.

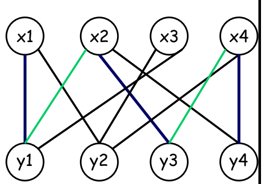
Path x1, y1, x2, y3, x4, y4 is augmenting.



Maximum Matching

If a matching M (in green) has an augmenting path, then we get a larger matching by swapping the edges on the augmenting path.





Hungarian Algorithm

The algorithm starts with any matching and constructs a tree via a breadth-first search to find an augmenting path.

If the search succeeds, then it yields a matching having one more edge than the original.
Then we search again (it most it happens is V/2) for a new augmenting path.
If the search is unsuccessful, then the algorithm terminates and must be the largest-size matching that exists.

What is the runtime complexity of the Hungarian algorithm?

Complexity of BFS – O(V+E)

We run it V/2 times

This, the runtime is O(V E).

Proof of Correctness

The algorithm clearly terminates, since we match one edge per step.

Suppose that there were another matching M1 that used more edges than M.

Overlap M and M1 - the result is a union of cycles and paths.

There is a path that have more M1 edges than from M.

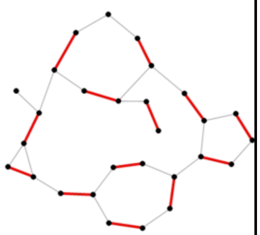
This path is an augmenting path. Contradiction.

Matching on Non-Bipartite Graphs

The Hungarian algorithm does not work on general graphs.

The problem is to find an augmenting path, that in general case might have an odd length cycle (called a blossom).

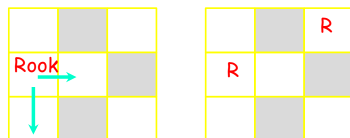
The blossom can be shrunk and the search restarted recursively.



The Blossom algorithm is due to Edmonds, 1965.

Rook Attack

This problem asks us to place a maximum number of rooks (they move horizontally and vertically) on a chessboard with some squares cut out (forbidden positions)



Rook Attack

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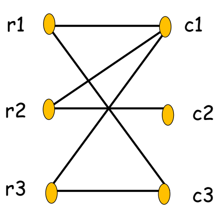

The number of non-attacking rooks equals the number of edges in a matching.

Here's What You Need to Know…

- Planar Graphs
- Kuratowski Theorem
- Graph Coloring
- Bipartite Graphs
- Bipartite Matching
- Hall Theorem