Plan

Some recap
Pigeonhole Principle
Pascal’s Triangle
Combinatorial Proofs
Manhattan Walk
Catalan Number

Permutations vs. Combinations

Subsets of \( r \) out of \( n \) distinct objects

\[
P(n,r) = \frac{n!}{(n-r)!} \quad \text{and} \quad \binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

Ordered Unordered

\# ways to arrange \( n \) symbols:

\( r_1 \) of type 1, \( r_2 \) of type 2, ..., \( r_k \) of type \( k \)

Multinomial Coefficient

\[
\binom{n}{r_1; r_2; \ldots; r_k} = \begin{cases} 
0, & \text{if } r_1 + r_2 + \ldots + r_k \neq n \\
\frac{n!}{r_1! r_2! \ldots r_k!}, & \text{otherwise}
\end{cases}
\]

Sequences with 20 G’s and 4 /’s

GG/G//GGGGGGGGGGGGGGGGGGG/G

In general, the \( j \)th pirate gets the number of G’s after the \( j-1 \)st / and before the \( j \)th /.

This gives a correspondence between divisions of the gold and sequences with 20 G’s and 4 /’s.
# ways to divide up the loot
= # sequences with 20 G's and 4 '/'s

\[
\binom{24}{4} = \binom{20 + 5 - 1}{5 - 1}
\]

Another interpretation
Number of different ways to throw \( n \) indistinguishable balls into \( k \) distinguishable bins:

\[
\binom{n+k-1}{k-1} = \binom{n+k-1}{n}
\]

How many nonnegative integer solutions to the following equations?

\[
x_1 + x_2 + \ldots + x_k = n
\]

Think of \( x_i \rightarrow y_i+1 \)

bijection with solutions to

\[
y_1 + y_2 + y_3 + \ldots + y_k = n-k
\]

Remember to distinguish between Identical / Distinct Objects

If we are putting \( n \) objects into \( k \) distinct bins.

| \( n \) objects are distinguishable | \( k^n \) |
| \( n \) objects are indistinguishable | \( \binom{n+k-1}{k-1} \) |
Pigeonhole Principle

If there are more pigeons than pigeonholes, then some pigeonholes must contain two or more pigeons.

Example:
Two people in Pittsburgh must have the same number of hairs on their heads.

Pigeonhole Principle

Problem: Prove that if seven distinct numbers are selected from \{1,2,3,...,11\}, some two of these numbers sum up to 12.

Pigeons: the chosen numbers
Holes: Falling in same hole should mean numbers sum up to 12
6 holes: \{(1,11), (2,10), (3,9), (4,8), (5,7), (6)\}
Place selected numbers in hole corresponding to the set containing it

Some two numbers fall in same hole, and thereby sum to 12.

Pigeonhole Principle

Problem:
Prove that among any \(n\) integers, there is a non-empty subset whose sum is divisible by \(n\).

Consider \(s_s = x_1 + ... + x_i\) modulo \(n\). How many \(s_s\)?
These are the \(n\) "pigeons"
Remainders modulo \(n\) belong to \(\{0, 1, 2, ..., n-1\}\).
If some remainder is 0, we are done.
If not, \((n-1)\) remainders \(\{1,2,...,n-1\}\). The "holes".

Exist \(s_i, s_k\), \(i < k\) such that \(n\) divides \(s_k - s_i\).
\(x_{i+1} + ... + x_k\) is our desired sum.

Pigeonhole Principle

Problem:
The numbers 1 to 10 are arranged in random order around a circle. Show that there are three consecutive numbers whose sum is at least 17.

What are pigeons?
And what are pigeonholes?
The numbers 1 to 10 are arranged in random order around a circle. Show that there are three consecutive numbers whose sum is at least 17.

Let \( S_1 = a_1 + a_2 + a_3 \), ..., \( S_{10} = a_{10} + a_1 + a_2 \). There are 10 pigeonholes.

Pigeons: \( S_1 + \ldots + S_{10} = 3(a_1 + a_2 + \ldots + a_{10}) = 3 \times 55 = 165 \)

Since 165 > 10 \( \times 16 \), there must exist a pigeonhole with at least 16 + 1 pigeons.

Actually, we’re using a generalization of the PHP. If \( x_1 + x_2 + \ldots + x_k > n \) then some \( x_i > n/k \)

Show that for some integer \( k > 1 \), \( 3^k \) ends with 0001.

Choose 10001 numbers: \( 3^1, 3^2, \ldots, 3^{10001} \).

\( 3^k \mod (10000), m < k \)

\( 3^k - 3^m = 0 \mod (10000), m < k \)

\( 3^m (3^k - 3^m - 1) = 0 \mod (10000), m < k \)

But 3 is relatively prime to 10000, so

\( 3^{k-m} = 1 \mod (10000) \)

\( 3^{k-m} = q \times 10000 + 1 \) ends with 0001.

Polynomials express choices and outcomes. Products of sum = sums of products.

\[
(\mathbb{1} + \mathbb{2} + \mathbb{3})(\mathbb{4} + \mathbb{5}) = \mathbb{7} + \mathbb{8} + \mathbb{9} + \mathbb{10} + \mathbb{11} + \mathbb{12} + \mathbb{13} + \mathbb{14} + \mathbb{15} + \mathbb{16}
\]

Now, to binomial theorem...

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
\]
There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!

\[(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2 + b^3t^1 + b^3t^2\]

Choice tree for terms of \((1+X)^3\)

Combine like terms to get \(1 + 3X + 3X^2 + X^3\)

What is the combinatorial meaning of those coefficients?

\((1+X)^n = c_0 + c_1X + c_2X^2 + \ldots + c_nX^n\)

What is a closed form expression for \(c_k\)?

\[(1 + X)^n \text{ \(n\) times} = (1 + X)(1 + X)(1 + X)(1 + X)\ldots(1 + X)\]

After multiplying things out, but before combining like terms, we get \(2^n\) cross terms, each corresponding to a path in the choice tree.

\(c_k\), the coefficient of \(X^k\), is the number of paths with exactly \(k\) \(X\)’s.

\[c_k = \binom{n}{k}\]

The Binomial Theorem

\[(1 + X)^n = \binom{n}{0}X^0 + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{n}X^n\]

Binomial Coefficients

binomial expression
The Binomial Formula

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

What is the coefficient of EMPTY in the expansion of \((E + M + P + T + Y)^5\) ?

What is the coefficient of EMP^3TY in the expansion of \((E + M + P + T + Y)^7\) ?

The number of ways to rearrange the letters in the word SYSTEMS

What is the coefficient of BA^3N^2 in the expansion of \((B + A + N)^9\) ?

The number of ways to rearrange the letters in the word BANANA

What is the coefficient of \(X_1 r_1 X_2 r_2 \ldots X_k r_k\) in the expansion of \((X_1 + X_2 + \ldots + X_k)^n\) ?

\[
\begin{cases} 
0, \text{if } r_1 + r_2 + \ldots + r_k \neq n \\
n! \\
r_1!r_2!\ldots r_k!
\end{cases}
\]

Multinomial coefficients

\[
\binom{n}{r_1; r_2; \ldots; r_k} = \begin{cases} 
0, \text{if } r_1 + r_2 + \ldots + r_k \neq n \\
\frac{n!}{r_1!r_2!\ldots r_k!}
\end{cases}
\]
The Multinomial Formula

\[
(X_1 + X_2 + \ldots + X_k)^n = \sum \binom{n}{r_1,r_2,\ldots,r_k} X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k}
\]

On to Pascal...

The binomial coefficients have so many representations that many fundamental mathematical identities emerge...

\[(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

Pascal’s Triangle:

The \(n^{th}\) row are the coefficients of \((1+X)^n\)

\[
\begin{align*}
(1+X)^0 &= 1 \\
(1+X)^1 &= 1 + 1X \\
(1+X)^2 &= 1 + 2X + 1X^2 \\
(1+X)^3 &= 1 + 3X + 3X^2 + 1X^3 \\
(1+X)^4 &= 1 + 4X + 6X^2 + 4X^3 + 1X^4
\end{align*}
\]

\[
\begin{align*}
\binom{n}{k} &= \binom{n-1}{k-1} + \binom{n-1}{k}
\end{align*}
\]

Blaise Pascal
1654

\[
\begin{align*}
1 + 2 + 1 \\
1 + 3 + 3 + 1 \\
1 + 4 + 6 + 4 + 1 \\
1 + 5 + 10 + 10 + 5 + 1 \\
1 + 6 + 15 + 20 + 15 + 6 + 1
\end{align*}
\]
Summing The Rows

\[ 2^n = \sum_{k=0}^{n} \binom{n}{k} \]

\[ \begin{align*}
1 & = 1 \\
1 + 1 & = 2 \\
1 + 2 + 1 & = 4 \\
1 + 3 + 3 + 1 & = 8 \\
1 + 4 + 6 + 4 + 1 & = 16 \\
1 + 5 + 10 + 10 + 5 + 1 & = 32 \\
1 + 6 + 15 + 20 + 15 + 6 + 1 & = 64
\end{align*} \]

Summing on 1st Avenue

\[ \sum_{k=0}^{n} \binom{n}{k} = \binom{n+1}{2} = \frac{n(n+1)}{2} \]

Summing on kth Avenue

\[ \sum_{m=1}^{n} \binom{m}{k} = \binom{n+1}{k+1} \]

Hockey-stick identity

Fibonacci Numbers

\[ \sum_{k=0}^{n-2} \binom{n-k}{k} = F_{n-1} \]

Sums of Squares

\[ \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \]
All these properties can be proved inductively and algebraically. But we will give combinatorial proofs.

How many ways we can create a size $k$ committee out of $n$ people?
LHS: By definition
RHS: We choose $n-k$ people to exclude from the committee.

How many ways we can create a size $k$ committee out of $n$ people?
LHS: By definition
RHS: Pick a person, say $n$.
There are committees that exclude person $x$
There are committees that include person $x$

LHS: We create a size $k$ committee, then we select a chairperson.
RHS: We select the chair out of $n$, then from the remaining $n-1$ choose a size $k-1$ committee.

LHS: Count committees of any size, one is a chair.
RHS: Select the chair out of $n$, then from the remaining $n-1$ choose a subset.
The art of combinatorial proof

\[
\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}
\]

Vandermonde’s identity

LHS: \(m\)-males, \(n\)-females, choose size \(k\).

RHS: Select a committee with \(j\) men, the remaining \(k-j\) members are women.

The art of combinatorial proof

\[
\binom{n+1}{k+1} = \sum_{m=k}^{n} \binom{m}{k}
\]

Hockeystick identity

LHS: The number of \((k+1)\)-subsets in \([1,2,\ldots,n+1]\)

RHS: Count \((k+1)\)-subsets with the largest element \(m+1\), where \(k \leq m \leq n\).

Manhattan walk

You’re in a city where all the streets, numbered 0 through \(x\), run north-south, and all the avenues, numbered 0 through \(y\), run east-west. How many [sensible] ways are there to walk from the corner of 0th st. and 0th avenue to the opposite corner of the city?

All these properties can be proved by using the Manhattan walking representation of binomial coefficients.

Manhattan walk on Pascal’s triangle

\[
\binom{x+y}{y} = \binom{x}{0} + \binom{x}{1} + \binom{x}{2} + \cdots + \binom{x}{y}
\]

level \(n\) (\(n\) total steps)

\[
\binom{n}{k} = \text{# Manhattan walks to this node}
\]
Manhattan walk on Pascal's Triangle

Manhattan walk

More Manhattan walk

Noncrossing Manhattan walk

14 such walks for n=4 (c.f. total # Manhattan walks = \binom{8}{4} = 70 )
Let's count # violating paths, that do cross the diagonal. Will do so by a bijection.

Find first step above the diagonal. “Flip” the portion of the path after that step.

How many sequences of n (‘s and n )’s are there such that every prefix has more (‘s than )’s?

Answer: \[ \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n} \]

The above is the n’th Catalan number. Miraculously pervasive:
- # permutations of \(1,2,\ldots,n\) that don’t have 3 term increasing subsequence
- # ways the numbers 1, 2, \ldots, 2n can be arranged in a 2-by-n rectangle so that each row and each column is increasing.
- Number of different ways a convex polygon with \(n+2\) sides can be cut into triangles by connecting vertices with straight lines.

Flip the portion of the path after the first edge above the diagonal.

Note: New path goes to \((n-1,n+1)\).

Claim (think about it):
Every Manhattan walk from \((0,0)\) to \((n-1,n+1)\) can be obtained in this fashion in exactly one way.

Thus, number of noncrossing Manhattan walks on \(n \times n\) grid =

\[ \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n} \]

Study Guide
- Pirates and Gold
- Binomial & Multinomial theorems
- Pigeonhole principle
- Combinatorial proofs of binomial identities
- Manhattan walks
- Catalan numbers