In the next 3-4 lectures we will learn some fundamental counting methods.

Addition and Product Rules
The Principle of Inclusion-Exclusion
Choice Trees
Permutations and Combinations
The Binomial Theorem
The Pigeonhole Principle
Diophantine Equations
Recurrences.
Generating Functions

If I have 14 teeth on the top and 12 teeth on the bottom, how many teeth do I have in all?

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Addition Rule
Let $A$ and $B$ be two disjoint finite sets

$$|A \cup B| = |A| + |B|$$

Addition of Multiple Disjoint Sets:
- Let $A_1, A_2, \ldots, A_n$ be disjoint, finite sets:

$$\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} |A_i|$$

Addition Rule
(2 Possibly Overlapping Sets)
Let $A$ and $B$ be two finite sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Inclusion-Exclusion
- If $A, B, C$ are three finite sets, what is the size of $(A \cup B \cup C)$?

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Inclusion-Exclusion
- If $A_1, A_2, \ldots, A_n$ are $n$ finite sets, what is the size of $(A_1 \cup A_2 \cup \ldots \cup A_n)$?

$$\sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Exercise: Prove this by induction!

Partition Method
To count the elements of a finite set $S$, partition the elements into non-overlapping subsets $A_1, A_2, A_3, \ldots, A_n$.

$$\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} |A_i|$$
Partition Method

$S = \text{all possible outcomes of one white die and one black die.}$

Partition $S$ into 6 sets:

$A_1 = \text{the set of outcomes where the white die is 1.}$

$A_2 = \text{the set of outcomes where the white die is 2.}$

$A_3 = \text{the set of outcomes where the white die is 3.}$

$A_4 = \text{the set of outcomes where the white die is 4.}$

$A_5 = \text{the set of outcomes where the white die is 5.}$

$A_6 = \text{the set of outcomes where the white die is 6.}$

Each of 6 disjoint set have size $6 = 36$ outcomes.

$S = \text{all possible outcomes where the white die and the black die have different values}$

Partition Method

$A_i \equiv \text{set of outcomes where black die says i and the white die says something else.}$

$S = \text{Set of all outcomes where the dice show different values.}$ $|S| = ?$

$|S| = \sum_{i=1}^{6} |A_i| = 5 + 5 = 30$

$|S| + |B| = \# \text{ of outcomes} = 36$

$|B| = 6$

$|S| = 36 - 6 = 30$

$B \equiv \text{set of outcomes where dice agree.}$

$S = \text{Set of all outcomes where the black die shows a smaller number than the white die.}$ $|S| = ?$

$L \equiv \text{set of all outcomes where the black die shows a larger number than the white die.}$

$|S| + |L| = 30$

It is clear by symmetry that $|S| = |L|$. Therefore $|S| = 15$

$\text{Difference Method}$

To count the elements of a finite set $S$, find two sets $A$ and $B$ such that $S$ and $B$ are disjoint and $S \cup B = A$ then $|S| = |A| - |B|$
Pinning Down the Idea of Symmetry by Exhibiting a Correspondence

Put each outcome in \( S \) in correspondence with an outcome in \( L \) by swapping color of the dice.

Each outcome in \( S \) gets matched with exactly one outcome in \( L \), with none left over.

Thus: \(|S| = |L|\)

Let \( f : A \rightarrow B \) Be a Function From a Set \( A \) to a Set \( B \)

\( f \) is injective (one-one) if and only if
\[
\forall x,y \in A, \ x \neq y \Rightarrow f(x) \neq f(y)
\]

\( f \) is surjective (onto) if and only if
\[
\forall z \in B \ \exists x \in A \ f(x) = z
\]

Let's Restrict Our Attention to Finite Sets

\( \exists \) injective (1-1) \( f : A \rightarrow B \) \( \Rightarrow |A| \leq |B| \)

\( \exists \) surjective (onto) \( f : A \rightarrow B \) \( \Rightarrow |A| \geq |B| \)

\( \exists \) bijective \( f : A \rightarrow B \) \( \Rightarrow |A| = |B| \)

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\( \exists \) bijective \( f \) means the inverse \( f^{-1} \) is well-defined

Correspondence Principle

If two finite sets can be placed into bijection, then they have the same size

It's one of the most important mathematical ideas of all time!

Question: How many n-bit sequences are there?

Each sequence corresponds to a unique number from 0 to \( 2^n - 1 \). Hence \( 2^n \) sequences.

\[
\begin{align*}
00000 & \leftrightarrow 0 \\
00001 & \leftrightarrow 1 \\
00010 & \leftrightarrow 2 \\
00011 & \leftrightarrow 3 \\
& \vdots \\
11111 & \leftrightarrow 2^{n-1}
\end{align*}
\]

\( S = \{ a,b,c,d,e \} \) has Many Subsets

\( \{a\}, \{a,b\}, \{a,d,e\}, \{a,b,c,d,e\}, \{e\}, \emptyset \)...

The entire set and the empty set are subsets with all the rights and privileges pertaining thereto
Question: How Many Subsets Can Be Made From The Elements of a 5-Element Set?

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

{ b c e } 1 means “TAKE IT” 0 means “LEAVE IT”

Each subset corresponds to a 5-bit sequence (using the “take it or leave it” code)

Let us define a map \( f : \mathbb{B} \rightarrow \mathcal{S} \)

For bit string \( b = b_1 b_2 b_3 ... b_n \), let \( f(b) = \{ a_i | b_i = 1 \} \)

Claim: \( f \) is injective

Any two distinct binary sequences \( b \) and \( b' \) have a position \( i \) at which they differ

Hence, \( f(b) \) is not equal to \( f(b') \) because they disagree on element \( a_i \)

Let \( S = \{ a_1, a_2, a_3, ... , a_n \}, T = \text{all subsets of } S \)

\( \mathbb{B} = \text{set of all } n \)-bit strings

For bit string \( b = b_1 b_2 b_3 ... b_n \), let \( f(b) = \{ a_i | b_i = 1 \} \)

Claim: \( f \) is surjective

- Let \( X \) be a subset of \( \{ a_1, a_2, ... , a_n \} \)
- Define \( b_i = 1 \) if \( a_i \in X \) and \( b_i = 0 \) otherwise.
- Note that \( f(b_1 b_2 ... b_n) = X \).

The number of subsets of an \( n \)-element set is \( 2^n \)

Let \( f : A \rightarrow B \) be a function from \( A \) to \( B \)

\( f \) is a 1 to 1 correspondence (bijection) iff \( \forall z \in \mathbb{B} \exists ! x \in A \text{ such that } f(x) = z \)

\( f \) is a \( k \) to 1 correspondence iff \( \forall z \in \mathbb{B} \exists k x \in A \text{ such that } f(x) = z \)

A \( k \) to \( 1 \) function

To count the number of horses in a barn, we can count the number of hoofs and then divide by 4

If a finite set \( A \) has a \( k \) to 1 correspondence to finite set \( B \), then \( |B| = |A|/k \)

I own 3 beanies and 2 ties. How many different ways can I dress up in a beanie and a tie?
A Restaurant Has a Menu With 5 Appetizers, 6 Entrees, 3 Salads, and 7 Desserts

How many items on the menu?
5 + 6 + 3 + 7 = 21

How many ways to choose a complete meal?
5 × 6 × 3 × 7 = 630

How many ways to order a meal if I am allowed to skip some (or all) of the courses?
6 × 7 × 4 × 8 = 1344

Leaf Counting Lemma
Let T be a depth-n tree when each node at depth 0 ≤ i ≤ n−1 has $P_i$ children
The number of leaves of T is given by:
$P_1 P_2 ... P_n$

Choice Tree
A choice tree is a rooted, directed tree with an object called a “choice” associated with each edge and a label on each leaf.

A choice tree provides a “choice tree representation” of a set S, if
1. Each leaf label is in S, and each element of S is some leaf label
2. No two leaf labels are the same

We will now combine the correspondence principle with the leaf counting lemma to make a powerful counting rule for choice tree representation.

Product Rule
Suppose every object of a set S can be constructed by a sequence of choices with $P_1$ possibilities for the first choice, $P_2$ for the second, and so on.

IF
1. Each sequence of choices constructs an object of type S
2. No two different sequences create the same object

THEN
There are $P_1 P_2 ... P_n$ objects of type S

How Many Different Orderings of Deck With 52 Cards?
What object are we making? Ordering of a deck

Construct an ordering of a deck by a sequence of 52 choices:
52 possible choices for the first card;
51 possible choices for the second card;
...
1 possible choice for the 52nd card.

By product rule: $52 × 51 × 50 × ... × 2 × 1 = 52!$

A permutation or arrangement of n objects is an ordering of the objects
The number of permutations of n distinct objects is n!

How many sequences of 7 letters are there?
$26^7$
(26 choices for each of the 7 positions)
How many sequences of 7 letters contain at least two of the same letter?

\[26^7 - 26 \times 25 \times 24 \times 23 \times 22 \times 21 \times 20\]

The Difference Principle
Sometimes it is easiest to count the number of objects with property \( Q \), by counting the number of objects that do not have property \( Q \).

If 10 horses race, how many orderings of the top three finishers are there?

\[10 \times 9 \times 8 = 720\]

Number of ways of ordering or arranging \( r \) out of \( n \) objects

- \( n \) choices for first place, \( n-1 \) choices for second place, \ldots
- \( n \times (n-1) \times (n-2) \times \ldots \times (n-(r-1)) \)

\[= \frac{n!}{(n-r)!}\]

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

\[52 \times 51\]

How many unordered pairs?

\[
\frac{52 \times 51}{2} \leftarrow \text{divide by overcount}
\]

Each unordered pair is listed twice on a list of the ordered pairs.

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

\[52 \times 51\]

How many unordered pairs?

\[
\frac{52 \times 51}{2} \leftarrow \text{divide by overcount}
\]

We have a 2:1 map from ordered pairs to unordered pairs. Hence #unordered pairs = (#ordered pairs)/2

A combination or choice of \( r \) out of \( n \) objects is an (unordered) set of \( r \) of the \( n \) objects

The number of \( r \) combinations of \( n \) objects:

\[\frac{n!}{r!(n-r)!} = \binom{n}{r}\]

Ordered Versus Unordered

How many ordered 5 card sequences can be formed from a 52-card deck?

\[52 \times 51 \times 50 \times 49 \times 48\]

How many orderings of 5 cards?

\[5!\]

How many unordered 5 card hands?

\[
\frac{52 \times 51 \times 50 \times 49 \times 48}{5!} = 2,598,960
\]

The number of subsets of size \( r \) that can be formed from an \( n \)-element set is:

\[\frac{n!}{r!(n-r)!} = \binom{n}{r}\]
Choosing position $i$ for the first 0 and then position $j$ for the second 0 gives the same sequence as choosing position $j$ for the first 0 and then position $i$ for the second 0. There are 2 ways of generating the same object!

How Many 8-Bit Sequences Have 2 0’s and 6 1’s?

Tempting, but incorrect: 8 ways to place first 0, times 7 ways to place second 0. 

Violates condition 2 of product rule (uniqueness)

Choosing position $i$ for the first 0 and then position $j$ for the second 0 gives the same sequence as choosing position $j$ for the first 0 and then position $i$ for the second 0. There are 2 ways of generating the same object!

How Many 8-Bit Sequences Have 2 0’s and 6 1’s?

1. Choose the set of 2 positions to put the 0’s. The 1’s are forced.

2. Choose the set of 6 positions to put the 1’s. The 0’s are forced.

Symmetry in the Formula

$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n}{n-r}$

“# of ways to pick $r$ out of $n$ elements”

“# of ways to choose the (n-r) elements to omit”

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How Many Hands Have at Least 3 Aces?

How many hands have exactly 3 aces?

$\binom{4}{3} \times \frac{48}{49} = \frac{4 \times 1176}{4512} = 4704$

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REVERSIBILITY
CHECK:
For each object
can I reverse
engineer the
unique sequence
of choices that
constructed it?

Scheme I
1. Choose 3 of 4 aces
2. Choose 2 of the remaining cards

\[
\begin{align*}
A \heartsuit A \spadesuit A \diamondsuit K \\
A \heartsuit A \spadesuit A \heartsuit \\
A \heartsuit A \spadesuit A \spadesuit K \\
A \heartsuit A \spadesuit A \heartsuit K
\end{align*}
\]
For this hand – you can’t reverse to a unique choice sequence.

Scheme II
1. Choose 3 out of 4 aces
2. Choose 2 out of 48 non-ace cards

\[
\begin{align*}
A \heartsuit A \spadesuit A \heartsuit K \\
A \heartsuit A \spadesuit A \spadesuit K
\end{align*}
\]
REVERSE TEST: Aces came from choices in (1) and others came from choices in (2)

DEFENSIVE THINKING
ask yourself:
Am I creating objects of the
right type?
Can I create every object of
this type?
Can I reverse engineer my
choice sequence from any
given object?

The three big mistakes people
make in associating a choice
tree with a set S are:
1. Creating objects not in S
2. Missing out some objects
   from the set S
3. Creating the same object two
different ways

A group of rabbits are playing outside
their individual burrows when they are
surprised by an eagle.
Each rabbit escapes down to a random
hole, one rabbit per hole.
What is the chance that no rabbit is in
its own individual hole?
How many ways are there for the
rabbits to reorganize while avoiding
their own hole?

\[
A_i = \text{set of permutations of six rabbits}
where i’th rabbit ends up in its hole
\]
Use inclusion-exclusion.

\[
|A_i| = 5!
\]
\[
|A_i \cap A_j| = 4!
\]
\[
|A_i \cap A_j \cap A_k| = 3!
\]

How many ways to
rearrange the letters in the
word "SYSTEMS"?

SYSTEMS
7 places to put the Y,
6 places to put the T,
5 places to put the E,
4 places to put the M,
and the S’s are forced

\[
7 \times 6 \times 5 \times 4 = 840
\]
Let's pretend that the S's are distinct:

\[ S_1 Y S_2 T E M S_3 \]

There are \( 7! \) permutations of \( S_1 Y S_2 T E M S_3 \).

But when we stop pretending we see that we have counted each arrangement of \( S_1 S_2 S_3 \) \( 3! \) times, once for each of \( 3! \) rearrangements of \( S_1 S_2 S_3 \).

\[
\frac{7!}{3!} = 840
\]

Arrange \( n \) symbols: \( r_1 \) of type 1, \( r_2 \) of type 2, \ldots, \( r_k \) of type \( k \)

\[
\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! r_2! \ldots r_k!}
\]

How many ways to rearrange the letters in the word "CarnegieMellon"?

\[
\frac{14!}{2! 3! 2!} = 3,632,428,800
\]

Multinomial Coefficients

\[
\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! r_2! \ldots r_k!}
\]

Four ways of choosing

We will choose 2 letters from the alphabet \( \{L, U, C, K, Y\} \)

1) \( \binom{5}{2} \) no repetitions, the order is NOT important
   \( LU = UL \)

2) \( P(5,2) \) no repetitions, the order is important
   \( LU \neq UL \)

4) Repetitions allowed, the order is NOT important
   \( \binom{5}{2} + \binom{5}{2} \cdot \binom{5}{2} = 35 \)

What if we choose 3-letter words from the alphabet \( \{L, U, C, K, Y\} \)

allow repetitions, the order is NOT important

\[
\binom{5}{3} = \frac{5!}{3! 2!} = 10
\]

What about 5-letter words?

20-letter words?
5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?

Sequences with 20 G’s and 4 /’s
GG/G/GGGGGG/G/GGGG/G
represents the following division among the pirates
1st pirate gets 2
2nd pirate gets 1
3rd gets nothing (this is allowed!)
4th gets 16
5th gets 1

Sequences with 20 G’s and 4 /’s
GG/G/GGGGGGGGGGGGGGG/G
In general, the jth pirate gets the number of G’s after the j-1st / and before the jth /.
This gives a correspondence between divisions of the gold and sequences with 20 G’s and 4 /’s.

Another interpretation
How many integer solutions to the following equations?

\[
x_1 + x_2 + x_3 + x_4 + x_5 = 20
\]
x_1, x_2, x_3, x_4, x_5 \geq 0

Think of \( x_k \) as being the number of gold bars that are allotted to pirate \( k \).

\[
\binom{24}{4} = \binom{20+5-1}{5-1}
\]

Another interpretation
How many different ways can k distinct pirates divide \( n \) identical, indivisible bars of gold?

\[
\binom{n+k-1}{k-1} = \binom{n+k-1}{n}
\]

Another interpretation
How many different ways to throw \( n \) indistinguishable balls into \( k \) distinguishable bins?

\[
\binom{n+k-1}{k-1} = \binom{n+k-1}{n}
\]

How many integer nonnegative solutions to the following equations?

\[
x_1 + x_2 + \ldots + x_r = m
\]

Think of \( x_k \) as being the number of gold bars that are allotted to pirate \( k \).

\[
\binom{m+r-1}{r-1} = \binom{m+r-1}{m}
\]

How many integer positive solutions to the following equations?

\[
x_1 + x_2 + x_3 + \ldots + x_k = n
\]
x_1, x_2, x_3, \ldots, x_k > 0

Think of \( x_k \) as being the number of gold bars that are allotted to pirate \( k \).

\[
y_1 + y_2 + y_3 + \ldots + y_k = n-k \binom{n-1}{k-1}
\]
y_1, y_2, y_3, \ldots, y_k \geq 0
Remember to distinguish between Identical / Distinct Objects

If we are putting \( n \) objects into \( k \) distinct bins.

<table>
<thead>
<tr>
<th>( n ) objects are distinguishable</th>
<th>( k^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n ) objects are indistinguishable</td>
<td>((n+k-1)\choose(k-1))</td>
</tr>
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</table>

Partition and Difference Methods
- Principle of Inclusion and Exclusion
- Correspondence Principle

If two finite sets can be placed into a 1-1 onto correspondence, then they have the same size.

Choice Tree
Product Rule
Two conditions
Reverse Test
Binomial & multinomial coefficient

Supplement: Testing your counting with Poker Hands

52 Card Deck, 5 card hands
- 4 possible suits: \( ♠️ ♥️ ♦️ ♣️ \)
- 13 possible ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A

Pair: set of two cards of the same rank
Straight: 5 cards of consecutive rank
Flush: set of 5 cards with the same suit

Ranked Poker Hands
- Straight Flush: a straight and a flush
- 4 of a kind: 4 cards of the same rank
- Full House: 3 of one kind and 2 of another
- Flush: a flush, but not a straight
- Straight: a straight, but not a flush
- 3 of a kind: 3 of the same rank, but not a full house or 4 of a kind
- 2 Pair: 2 pairs, but not 4 of a kind or a full house
- A Pair

Straight Flush
- 9 choices for rank of lowest card at the start of the straight
- 4 possible suits for the flush
- \[ \frac{9 \times 4 \times 36}{52 \choose 5} = \frac{36}{2,598,960} \approx 1 \text{ in } 72,193 \text{ chance} \]

4 of a Kind
- 13 choices of rank
- 48 choices for remaining card
- \( 13 \times 48 = 624 \)
- \[ \frac{624}{5 \choose 5} = \frac{624}{2,598,960} = 1 \text{ in } 4,165 \]

Flush
- 4 choices of suit
- \( \left\{ \begin{array}{l} 13 \choose 5 \end{array} \right\} \) choices of cards
- “but not a straight flush…” - 36 straight flushes
- \[ \frac{5,112}{5 \choose 5} = 5112 \text{ flushes} \]

Straight
- 9 choices of lowest card
- 4\(^{th}\) choices of suits for 5 cards
- \( 9 \times 1024 = 9,216 \)
- “but not a straight flush…” - 36 straight flushes
- \[ \frac{9,180}{5 \choose 5} = 9,180 \text{ straights} \]
<table>
<thead>
<tr>
<th>Hand</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Straight Flush</td>
<td>36</td>
</tr>
<tr>
<td>Four of a Kind</td>
<td>624</td>
</tr>
<tr>
<td>Full House</td>
<td>3,744</td>
</tr>
<tr>
<td>Flush</td>
<td>5,112</td>
</tr>
<tr>
<td>Straight</td>
<td>9,180</td>
</tr>
<tr>
<td>Three of a Kind</td>
<td>54,912</td>
</tr>
<tr>
<td>Two Pair</td>
<td>123,552</td>
</tr>
<tr>
<td>One Pair</td>
<td>1,098,240</td>
</tr>
<tr>
<td>Nothing</td>
<td>1,302,540</td>
</tr>
<tr>
<td></td>
<td>2,598,960</td>
</tr>
</tbody>
</table>