15-251 : Great Theoretical Ideas In Computer Science

Fall 2014

Assignment 6

Due: Thursday, Oct. 16, 2014 11:59 PM

Name: ____________________

Andrew ID: ____________________

Question:	1	2	3	4	5	Total
Points:	20	20	30	15	15	100
Score:						

0. Warmup

(a) A connected, planar graph has nine vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4, and 5. How many edges does the graph have? How many faces does it have?

(b) A planar connected graph G has 10 nodes each of degree 4. Is it possible that G is bipartite?

(c) Construct the labeled tree corresponding to the given Prufer sequence $1, 3, 2, 3, 5$.

(d) How many spanning trees does K_5 have?

(e) Given these preferences:

Men's preferences					Women's preferences				
1	2	4	1	3	1	2	1	4	3
2	3	1	4	2	2	4	3	1	2
3	2	3	1	4	3	1	4	3	2
4	4	1	3	2	4	2	1	4	3

Find a stable matching by executing the stable marriage algorithm. Remember, man proposes, woman disposes.

(f) Prove that every tree is bipartite.

(g) A connected graph on n vertices has exactly n edges. How many cycles does the graph have?

(h) Prove that the vertices of a graph of maximum degree d can be colored with $d + 1$ colors so that the endpoints of every edge have distinct colors. Give an example to show that sometimes $d + 1$ colors are necessary.

1. Coloring

Let G be a simple undirected graph without any cycles of length greater than 251.

(15) (a) Prove that the vertices of G can be colored with 251 colors such that any pair of adjacent vertices receive distinct colors.

Hint: DFS tree.

Solution:

(5) (b) Give an example to show this is tight, i.e., a graph G with no cycles of length greater than 251 that cannot be colored with 250 colors.

Solution:

2. Matchings

Suppose G is a simple undirected graph that barely misses out on having a perfect matching. That is, while G does not have a perfect matching, adding an edge between *any* pair of nonadjacent vertices in G creates a graph that does have a perfect matching.

Let U be the set of vertices in G with degree $|V(G)|-1$ (i.e. each vertex in U is adjacent to every other vertex in G).

(20) (a) Prove that the subgraph G' induced by $V(G) \setminus U$ consists of disjoint complete graphs.

Hint: Prove by contradiction, based on existence (justify this!) of four vertices $\{a, b, c, d\}$ such that a is adjacent to b, c but not d, and b is not adjacent to c.

Solution:

3. Barely non-planar

(10) (a) Let G be a planar graph and e be an arbitrary edge of G. Argue why there is a planar drawing of G such that e occurs on the external (unbounded) face of the drawing.

(20) (b) Let G be a non-planar graph with all degrees at least 3 such that every proper subgraph of G is a planar graph. Prove that G must be 3-connected. That is, show that it is impossible to disconnect G by the removal of at most two vertices.

<u>Hint</u>: Suppose that $G = G_0 \cup G_1$ with $V(G_0) \cap V(G_1) = \{a, b\}$, $|V(G_i)| \geq 3$. Use planarity of $G_i +$ (any a-b path in G_{1-i}) to derive a contradiction.

Solution:

4. Counting Trees

(15) (a) How many labeled trees on n vertices have exactly three vertices of degree 1? You must give a closed-form for the answer in terms of n.

Solution:

5. Pathfinder

A forest is a graph whose components are all trees.

(15) (a) Let G be a forest with exactly $2k$ vertices of odd degree. Prove that there exist k edge-disjoint paths $P_1, P_2, ..., P_k$ in G such that $E(G) = E(P_1) \cup E(P_2) \cup ... \cup E(P_k)$.

Solution: