15-251 : Great Theoretical Ideas In Computer Science

Fall 2014

Assignment 5

Due: Wednesday, Oct. 8, 2014 11:59 PM

Name: ______________________________

Andrew ID: ______________________________

Question:	1	2	3	4	Total
Points:	30	15	30	25	100
Score:					

0. Warmup

(a) What is the probability that **exactly 2 heads** appear in **4** independent tosses of a fair coin?

(b) What is the conditional probability that **exactly 2 heads** appear in **4** independent tosses of a fair coin, *given that the first toss lands* **tails**?

(c) In the hope of having a dry outdoor wedding, John and Mary decide to get married in the desert, where the average number of rainy days per year is 10. Unfortunately, the weather forecaster is predicting rain for tomorrow, the day of John and Marys wedding. Suppose that the weather forecaster is not perfectly accurate: If it rains the next day, 90% of the time the forecaster predicts rain. If it is dry the next day, 10% of the time the forecaster still (incorrectly) predicts rain. Given this information, what is the probability that it will rain during John and Marys wedding?

(d) Circle True or False (you don't need to justify your answer).

For all events A, B,

1. $\Pr[\overline{A}|B] = 1 - \Pr[A|B]$
2. $\Pr[A|\overline{B}] = 1 - \Pr[A|B]$
3. $\Pr[\overline{A}|\overline{B}] = 1 - \Pr[A|B]$
4. If events A and B are independent, then so are $\overline{A}$ and B.

   (As usual $\overline{A}$ (resp. $\overline{B}$) denotes the complement of the event A (resp. B).)

(e) Suppose we have a damaged die in which the 6 shows like a 3. Assume we throw the die twice, the throws being independent, and record values A and B.

1. What is the expected value of $A + B$?
2. What is the expected value of the *product* AB?
3. What is the expected value of B given $A = 2$?

1. Interview problems

(15) (a) Suppose we have an interval of length 1. David and Tae are playing a game. They throw two darts at the interval independently and uniformly at random, i.e., the position of each dart lands corresponds to a random variable distributed uniformly on the interval $(0, 1)$. The two darts divide our interval into three segments. David wins iff. these three segments form a triangle. What's the probablty that David wins?

Solution:

(15) (b) 15251 people line up to enter a stadium for a concert, where Patrick is the first person in the line and Klaas is the last person in the line. Everyone has a ticket with assigned seat. However, Patrick has lost his ticket and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of unoccupied seats at random otherwise. What is the probability that Klaas gets to sit in his assigned seat?

Solution:

2. Testing of vaccines

A pharmaceutical company has developed a potential vaccine against the H1N1 flu virus. Before any testing of the vaccine, the developers assume that with probability 0.5 their vaccine will be effective and with probability 0.5 it will be ineffective. The developers do an initial laboratory test on the vaccine. This initial lab test is only partially indicative of the effectiveness of the vaccine, with an accuracy of 0.6. Specifically, if the vaccine is effective, then this laboratory test will return "success" with probability 0.6, whereas if the vaccine is ineffective, then this laboratory test will return "failure" with probability 0.6.

(5) (a) What is the probability that the laboratory test returns "success"?

Solution:

(10) (b) What is the probability that the vaccine is effective, given that the laboratory test returned "success"?

Solution:

3. I am the master of conditional expectation

(15) (a) Victor and Venkat bought a supercomputer to solve 15-251 problems. However, in all computers shipped from the manufacture, 99% of them are "good" and 1% of them are "bad". The good computers have probability 1/30 of failing after solving a 15-251 problem. The bad computers have probability 1/2 of failing after solving a 15-251 problem.

Let T denote the number of 15-251 problems solved until the supercomputer fails, what is $E[T]$?

Solution:

(15) (b) Andy is lost somewhere in Pittsburgh, and he decides to use a randomized algorithm to find his way home. Initially he has two directions. With probability 1/2 Andy goes to the right and probability 1/2 to the left. If he goes to the right, then he will wander around for 3 minutes and then return to its initial position. If he goes to the left, then with probability 1/3 , he will arrive at Forbes Avenue (where he knows the way home) after 2 minutes of traveling, and with probability 2/3 he will return to his initial position after 5 minutes of traveling.
Let T denote the number of minutes Andy takes to arrive at Forbes Avenue. What is $E[T]$?

Solution:

4. Kittens with probability

(10) (a) Peter's kittens are very fascinated by threads. He has given them n threads to play with. The kittens start playing the following game: At each step, if k is the number of threads remaining (initially $k = n$), then a kitten chooses two of the $2k$ free ends randomly (each pair of free ends is chosen with equal probability among all pairs), and ties them together. Two things can happen:

1. This operation merges two distinct threads into one.
2. The kitten ends up tying two ends of the same thread together, into a loop. Kittens like loops! If this happens, the kitten takes the loop, and starts playing with it.

In any case, the number of threads remaining to play with decreases by 1. The game ends when there are no more threads to play with. Find the expected number of loops created at the end of this game.

Solution:

(15) (b) Peter has decided to put 100 of his kittens for adoption. He has lined up the kittens in a row against a wall in a hallway of the house such that the door of the hallway is at the right end of the line of kittens. He has gathered 100 friends to adopt the kittens. Each friend i has already selected a number X_i from 1 to 100 randomly (all numbers with equal probability), and is determined to take home the kitten standing at position X_i in the line.

Peter let his friends into the hallway one at a time. They keep walking down the hallway until one of two things happens:

- They reach their favorite kitten. At that point, they sit down and begin to play with that kitten.
- Their path is blocked. Since the hallway is rather narrow, any person sitting and playing with a kitten will block anyone from going past them. Instead, the person who has to choose a kitten sits down and begins to play with the kitten immediately before the one whose new owner is blocking his path.

If the kitten that is closest to the door is being played with, the rest of his friends still outside will not be able to enter the hall at all, and will have to leave empty handed. What is the most likely (note: "most likely" is **not** "expected value") number of kittens that will be adopted?

Solution: