

**15-251 : Great Theoretical Ideas In Computer Science****Fall 2014****Assignment 2**

Due: Thursday, Sep. 11, 2014 11:59 PM

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Question:	1	2	3	4	Total
Points:	30	20	25	25	100
Score:					

## 0. Warmup

- (a) Define an axiomatic system as follows: The universe is the set of lists of natural numbers.

$[]$  is an axiom.

For any  $x \in \mathbb{N}$ , from a list  $\ell$ , derive  $x :: \ell$ , the list whose first element is  $x$ , and the rest of which is  $\ell$ .

Prove soundness and completeness of this system with the truth concept “ $\ell$  is a list of natural numbers.” In other words, prove that every element of the universe is a theorem of this system.

- (b) Given a propositional formula  $P$ , prove that  $P$  is valid iff  $\neg P$  is not satisfiable.

- (c) Show that  $\forall x \exists y P(x, y) \rightarrow \forall z P(z, z)$  is not valid.

- (d) Decide whether or not the following formulae are true for each of these universes:  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ .

1.  $\exists x(x^2 = 2)$ .
2.  $\forall x \exists y(x^2 = y)$ .
3.  $\forall y \exists x(x^2 = y)$ .
4.  $\forall x(x \neq 0 \rightarrow \exists y(xy = 1))$ .

**1. Axiomatic Systems (30 points)**

Define an axiomatic system as follows:

The universe is the set of ordered pairs of natural numbers.

$(0, 0)$  is an axiom.

There are three deduction rules:

1. From  $(a, b)$ , deduce  $(a, b + 1)$ .
2. From  $(a, b)$ , deduce  $(a + 1, b + 1)$ .
3. From  $(a, b)$ , deduce  $(a + 2, b + 1)$ .

- (10) (a) Characterize the set of all theorems in this axiomatic system.

**Solution:**

- (10) (b) Prove the soundness of the system with the truth concept defined by your characterization.

**Solution:**

- (10) (c) Prove the completeness of the system with the truth concept defined by your characterization.

**Solution:**

**2. Propositional Logic (20 points)**

- (15) (a) Given a boolean formula  $F$  in the language of propositional logic, with variables  $x_1, x_2, \dots, x_n$ , write a formula  $G$  which is a tautology if and only if  $F$  is satisfiable. Prove that your formula has this property.

**Solution:**

- (5) (b) Prove that  $\rightarrow$  distributes over  $\wedge$  – that is, prove that

$$p \rightarrow (q \wedge r) \leftrightarrow (p \rightarrow q) \wedge (p \rightarrow r)$$

is a tautology.

**Solution:**

## 3. First Order Logic (25 points)

- (5) (a) Negate the following formula in first-order logic:

$$\forall x \forall \varepsilon (\varepsilon > 0 \rightarrow \exists \delta (\delta > 0 \wedge ((\forall y (|y - x| < \delta)) \rightarrow (|f(y) - f(x)| < \varepsilon))).$$

<b>Solution:</b>
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- (b) For the following parts, take the universe to be the set of all finite binary strings. The constant symbols are 0, 1, and  $\varepsilon$  (the empty string). There is one function symbol  $cat(x, y)$ , which gives the concatenation of  $x$  and  $y$  as strings. There is one relation symbol  $=$ , such that  $x = y$  is true iff  $x$  and  $y$  are equal as strings. Write, in the language of first-order logic, a statement which is equivalent to the given relation. You may use a relation that you have already defined in a previous part if you wish. Example:  $isOdd(x)$ , which determines if the number whose binary representation is  $x$  is odd. Then,  $isOdd(x) \leftrightarrow \exists a (x = cat(a, 1))$ .

- (4) i.
- $isEven(x)$
- , which determines if the number whose binary representation is
- $x$
- is even.

<b>Solution:</b>
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- (4) ii.
- $isSubstring(x, y)$
- , which determines if the string
- $x$
- is a substring of
- $y$
- .

<b>Solution:</b>
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- (4) iii.
- $isPowerOf2(x)$
- , which determines if the number whose binary representation is
- $x$
- is a power of 2.

<b>Solution:</b>
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- (4) iv.
- $isPalindrome(x)$
- , which determines if the string
- $x$
- is a palindrome.

<b>Solution:</b>
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- (4) v.
- $isEqual(x, y)$
- , which determines if the numbers represented by
- $x$
- and
- $y$
- are equal.

<b>Solution:</b>
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## 4. Miscellany (25 points)

The subtree rooted at a node  $v$  of a rooted tree is the set of all nodes  $x$  such that any path from  $x$  to the root of the tree goes through  $v$ . To attach a rooted tree  $T$  to a node  $v$  of another tree, draw an edge from the root of  $T$  to  $v$ . Note that a rooted tree need not be a binary tree – the branching factor need not even be fixed.

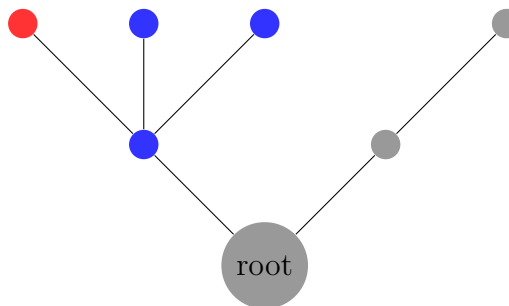
- (25) (a) Suppose we have a rooted tree, which we can modify according to the following rule:

Remove a leaf of the tree. If its parent is the root of the tree, continue. Otherwise, take 251 copies of the subtree rooted at the parent of the removed leaf (after the removal of the leaf, rather than before), and attach them to the grandparent of the removed leaf.

Prove that regardless of the initial configuration of the tree, you can eventually turn the tree into the one-vertex tree that is just the root by applications of this rule.

**Solution:**

**Example.** Here is a rooted tree, before any change has been made. The red node is the leaf that is going to be removed, and the blue nodes are the subtree rooted at the parent of the leaf to be copied.



This is a tree after the leaf has been removed. It has one extra copy (in green) of the relevant subtree added (we don't want to draw 251 copies).

