Contact mechanics and adhesion

- Surface and adhesion forces
- Derjaguin approximation
- Indentation forces

Consider the energy per unit area in the interaction of two planar surfaces:

\[ w(0) = -2\pi \sigma \frac{1}{(n-2)(n-3)} D^{n-3} \]
For a thin sheet of molecules with unit area and thickness \(d_2\), at a distance \(r\) away from another surface (of large area), the interaction energy is:

\[
W(r) = -\frac{2\pi \sigma q^2}{(n-2)(n-3)} r^{n-3}
\]

interaction energy for a sheet of unit area and thickness \(d_2\)

number of molecules in a sheet of unit area and thickness \(d_2\)

For two surfaces (integration over all sheets \(d_2\)):

\[
W(D) = -\frac{2\pi \sigma q^2}{(n-2)(n-3)} \int_0^\infty \frac{d_2^2}{2^{n-3}} = \frac{2\pi \sigma q^2}{(n-2)(n-3)(n-4)} D^{n-4}
\]

For \(n = 6\)

\[
W(D) = -\frac{\pi \sigma q^2}{12 D^2}
\]
Interaction energy per unit area for two planar surfaces can be used to describe the van der Waals' force between macroscopic bodies. Consider the torque between sphere and flat surface:

$$F(D) = -\frac{\partial W(D)}{\partial D} = -\frac{4\pi \varepsilon_0 \varepsilon^2 R}{(n-2)(n-3)(n-4)} \Delta n^3$$

Compare with

$$F(D)_{\text{sphere}} = 2\pi R W(D)_{\text{planes}}$$ per unit area.

So far, we have dealt only with additive potentials $W(r) = -C/r^n$.

The obtained relationship is however valid for any type of force law.
Consider two large spheres of radii $R_1$ and $R_2$, separated by small distance $D$

If $R_1 \gg D$ and $R_2 \gg D$, the force between the spheres can be obtained by integrating the force between the small circular regions $2\pi x \, dx$ on $z_1$ shown above.

$$ F(D) = \int_{z=0}^{z_1} 2\pi x \, dx \cdot f(z) $$

$f(z)$: normal force per unit area between two flat plates.
Using chords theorem:
\[ x^2 = 2R_1z_1 = 2R_2z_2 \]

\[ Z = D + z_1 + z_2 = D + \frac{x^2}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \]

Thus:
\[ dZ = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) dx \]

and
\[ F(D) \propto \int \frac{2\pi}{\left(\frac{R_1R_2}{R_1+R_2}\right)} f(z) dZ = \]
\[ = 2\pi \left(\frac{R_1R_2}{R_1+R_2}\right) W(0) \]

Notice: independent energy per unit area for two spheres of potential separation D we use!

Deviapun approximation: valid for any force law, provided that the separation distance D is much longer than the radii of the spheres.
Some special cases:

- $R_2 \gg R_1$
  \[ F(D) = 2\pi R_1 W(D) \]

- $R = R_1 = R_2$ (two equal spheres)
  \[ F(D) = \pi R W(D) \]

- Two spheres in contact ($D = 0$)
  
  The value of $W(D)$ can be associated with the surface free energy per unit area of the surface $\gamma$

  \[ W(D) = 2\gamma \]

  \[ F(D) = F_{ad} = \frac{4\pi \gamma R_1 R_2}{R_1 + R_2} \]

  Adhesion force between two spheres.
**Surface free energy**

Consider the cost of separating a body into two pieces

\[ W_{11} \rightarrow \text{work of cohesion} \]

\[ \Gamma_1 = \frac{1}{2} W_{11} \quad \text{Surface free energy (per unit area)} \]

We can calculate \( W_{11} \) (and thus \( \Gamma_1 \)) based on our simple pairwise summation of interaction energies between identical media.

\[ W = -\frac{A}{(2\pi D^2)} \]

Work required to separate two pieces by \( D \):

\[ \Delta W = \frac{A}{12\pi} \left( \frac{1}{D_0^2} - \frac{1}{D^2} \right) \]

\[ \Delta W = \frac{A}{12\pi D_0^2} \left( 1 - \frac{D_0^2}{D^2} \right) \]

Separation to infinity \( D = \infty \)

\[ \Delta W = W_{11} = \frac{A}{24\pi D_0^2} = 2\Gamma \]

\[ \Gamma = \frac{A}{24\pi D_0^2} \]

Surface free energy as a function of Hamaker constant.
Indentation

Flat cylindrical indenter

\[ E_1, E_2 - \text{Sample and indenter Young's moduli} \]
\[ R - \text{Indenter radius} \]
\[ \nu_1, \nu_2 - \text{Poisson coefficients} \]

For \( E_2 \gg E_1 \),
\[ F = \frac{2E_1R}{1-\nu_1^2} d \]

\[ \text{if } E_1 \text{ and } E_2 \text{ are comparable} \]
\[ F = \frac{2}{\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}} R \]
\[ d \sim F \]
Conical indenter

\[ F = \tan \alpha \cdot \frac{2E}{\pi(1-\nu^2)} d^2 \]

\[ d \approx F^{1/2} \]

Sphere-sphere contact (Hertz)
\[ d = d_1 + d_2 \quad \text{total deformation} \]

\[ K_1 = \frac{1 - \nu_1^2}{E_1} \quad K_2 = \frac{1 - \nu_2^2}{E_2} \]

\[ K_{\text{eff}} = K_1 + K_2 \]

\[ R_{\text{eff}} = \frac{R_1 \cdot R_2}{R_1 + R_2} \]

**Contact radius**: \[ n^2 = \frac{3 \pi}{4} \cdot R_{\text{eff}} \cdot K_{\text{eff}} \cdot F \]

\[ d = \frac{r^2}{R_{\text{eff}}} \]

If \( r_1 \to \infty \), \( R_{\text{eff}} = R_2 \)

- \( d \) - indentation
- \( r \) - contact radius
indentation vs load in Hertz model:

\[ d^3 = \frac{r^6}{K_{eff}} = \frac{3\pi}{4} \frac{K_{eff}^2}{r_{eff}} \cdot F^2 \]

\[ d \approx F^{2/3} \]

distribution of pressure in contact region

\[ P(r, x) = \frac{2F}{\pi^2 K_{eff} r_{eff}} \sqrt{1-x^2} \]

where \( r \) - contact radius

\[ x = \frac{r}{r_{eff}} \]

\( x \) - normalized distance from the center

\( s \) - distance from the center.
JKR (Johnson-Kendall-Roberts) description of contact: Indentation + Adhesion

\[ r^3 = \frac{3\pi}{4} \left( \frac{K_{eff} \alpha}{F + q + \sqrt{q^2 + 2qF}} \right) \]

Additional load due to adhesion

\[ q = 3\pi K_{eff} W_{1,2} \]

\( W_{1,2} \) - work of adhesion

If external force \( F = 0 \)

\[ n_0 = \left( \frac{K_{eff}}{K_{eff}} \right) \sqrt{3} \]

Indentation:

\[ d = -\frac{r^2}{K_{eff}} + \frac{2}{3} \frac{r_0^{3/2}}{K_{eff}} r^{1/2} \]

For negative loads \( F < 0 \)

contact radius can be still positive, until

\[ \sqrt{q^2 + 2qF} > 0 \]

This means that a neck connecting the bodies in contact is formed.
The force at which the neck breaks can be determined from the condition:

\[ q^2 + 2qF_s = 0 \]

\[ F_s = -\frac{q}{2} = -\frac{3}{2} \pi \text{RSHT W12} \]

Neck radius upon breaking:

\[ r_s^2 = \frac{3\pi}{4} \text{RSHT Kett} \left( -\frac{q}{2} + q \right) = \frac{3\pi}{4} \text{RSHT Kett} \frac{q}{2} \]

Since \( r_0^2 = \frac{3\pi}{4} \text{RSHT Kett} \cdot 2q \)

\[ r_s = \frac{r_0}{\sqrt{4}} \times 0.6 \frac{r_0}{2} \]

Distribution of pressure:

\[ P(r, x) = \frac{2F}{\pi r^2 \text{RSHT Kett}} \sqrt{r^2 - \frac{\sqrt{\pi/2} w_l^2 \text{Kett} \cdot r}{\sqrt{1-x^2}}} \]