Linear Harmonic oscillator

Mechanical model: mass m on a spring characterized by a spring constant k

Elastic restoring force $F_s = -kx$ is balanced according to Newton's second law

$$F_s = m\underbrace{a}_{\text{acceleration}}$$

$$m\ddot{x} = -kx$$

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega_0^{\ 2} x = 0$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}$$

free natural angular frequency

We will show that such system oscillates with amplitude A and angular frequency ω_0.

Basic facts about second order linear differential equations:

1. Solutions $x = x(t; C_1, C_2)$ will have two constants dependent on initial conditions
2. If $x_1(t)$ is a solution then $Cx_1(t)$ is also a solution.
3. If $x_1(t)$ and $x_2(t)$ are solutions then $x_1(t) + x_2(t)$ as well as any linear combination $C_1x_1(t) + C_2x_2(t)$ is also a solution.

To solve the equation of motion

$$\ddot{x} + \omega_0^{\ 2} x = 0$$

we multiply both sides by $2\dot{x}$

$$2\dot{x}\ddot{x} = -2\omega_0^{\ 2} x\dot{x}$$

which allows us to immediately carry out the first integration:

$$\dot{x}^2 = -\omega_0^{\ 2} x^2 + C$$

We determine the first integration constant *C* by observing that at a full "swing", the oscillator position is equal to its amplitude, *x=A,* and its velocity is zero (turning point of oscillation) $\dot{x}=0$. Thus $C=\omega_0^{\,2}A^2$ and the equation to be integrated further becomes:

$$\dot{x}^2=\omega_0^{\,2}(A^2-x^2)$$

Separation of variables gives

$$\int\frac{dx}{\sqrt{A^2-x^2}}=\omega_0\int dt$$

and after integration:

$$\sin^{-1}\left(\frac{x}{A}\right)=\omega_0 t+\phi$$

or in a more familiar form:

$$x=A\sin(\omega_0 t+\phi)$$

ω_0 is the natural *angular frequency*.

$\omega_0=2\pi f_0$, where f_0 is the natural *frequency.*

Oscillation period $T_0=\dfrac{1}{f_0}$.

Alternative solution method uses trial exponential function

$x=\exp(\lambda t)$, and our task is to determine the constant λ.

Substituting into the equation to be solved we obtain

$$\lambda^2\exp(\lambda t)+\omega_0^{\,2}\exp(\lambda t)=0$$

from which it is immediately clear that

$$\lambda^2=-\omega_0^{\,2}$$

and thus

$$\lambda=\pm i\,\omega_0 \quad \text{where} \quad i=\sqrt{-1}$$

Thus the general solution has to be of the form

$$x = A_1 \exp(+i\omega_0 t) + A_2 \exp(-i\omega_0 t)$$

(remember: we need *two* constants).

Recall Euler's formula

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

Since A_1 and A_2 are complex quantities, they may be inconvenient to use, and it is more informative to write the solution as:

$$x = A\exp[i(\omega_0 t + \phi)]$$
$$= A\cos(\omega_0 t + \phi) + iA\sin(\omega_0 t + \phi)$$

Energy considerations

Consider solution

$$x = A\sin(\omega_0 t + \phi)$$

The velocity v is equal to

$$\dot{x} = A\omega_0 \cos(\omega_0 t + \phi)$$

and thus the kinetic energy

$$K = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2{\omega_0}^2\cos^2(\omega_0 t + \phi) = K_0\cos^2(\omega_0 t + \phi)$$

where the maximum kinetic energy is equal to

$$K_0 = \frac{1}{2}mA^2{\omega_0}^2 = \frac{1}{2}kA^2$$

The potential energy – work done by applied force displacing the system from 0 to x

$$U(x) = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

Substituting x

$$U(x) = \frac{1}{2}kA^2 \sin^2(\omega_0 t + \phi) = U_0 \sin^2(\omega_0 t + \phi)$$

where U_0 is the maximum potential energy (for $x=A$)

$$U_0 = \frac{1}{2}kA^2$$

The average values over one oscillation period are calculated using the definition

$$\langle f \rangle = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t)dt$$

Thus:

$$\langle U \rangle = \frac{\int_0^T U dt}{\int_0^T dt} = \frac{\int_0^T U_0 \sin^2(\omega_0 t + \phi)}{T} = \frac{1}{2}U_0 = \frac{1}{2}kA^2$$

and

$$\langle K \rangle = \frac{1}{2}K_0 = \frac{1}{2}kA^2 .$$

In conclusion, the average values of kinetic and potential energy per oscillation cycle are equal to

$$\langle U \rangle = \langle K \rangle = \frac{1}{2}\langle E \rangle$$

Damped harmonic oscillator

This time, we introduce the additional force, which will dissipate the energy.

$$F_d = -bv = -b\dot{x}$$

The equation of motion gains one more term:

$m\ddot{x} + b\dot{x} + kx = 0$

Denote:

$\gamma = \frac{b}{2m},\ \omega_0{}^2 = \frac{k}{m}$

$\ddot{x} + 2\gamma\dot{x} + \omega_0{}^2 x = 0$

Use exponential trial function

$x = \exp(\lambda t) \quad \dot{x} = \lambda\exp(\lambda t) \quad \ddot{x} = \lambda^2\exp(\lambda t)$

$\exp(\lambda t)[\lambda^2 + 2\gamma\lambda + \omega_0{}^2] = 0$
since

$\exp(\lambda t) \neq 0$
$\lambda^2 + 2\gamma\lambda + \omega_0{}^2 = 0$

$\lambda_1 = -\gamma + \sqrt{\gamma^2 + \omega_0{}^2}$

$\lambda_2 = -\gamma - \sqrt{\gamma^2 + \omega_0{}^2}$

$x(t) = A_1\exp(\lambda_1 t) + A_2\exp(\lambda_2 t)$
$x(t) = \exp(-\gamma t)[A_1\exp(+t\sqrt{\gamma^2 - \omega_0{}^2}) + A_2\exp(-t\sqrt{\gamma^2 - \omega_0{}^2})]$

Depending on the sign of the expression under the root, there are three possible cases:
Underdamped
$\gamma^2 - \omega_0{}^2 < 0$ λ_1 and λ_2 are imaginary: oscillating solutions.
Overdamped
$\gamma^2 - \omega_0{}^2 > 0$ λ_1 and λ_2 are both real
Critically damped
$\gamma^2 = \omega_0{}^2$

Underdamped oscillator

$$\omega_1 = \sqrt{\omega_0^{\,2} - \gamma^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

$$x(t) = \exp(-\gamma t)[A_1 \exp(i\omega_1 t) + A_2 \exp(i\omega_1 t)]$$

which recalling Euler's formula becomes

$$x(t) = \exp(-\gamma t)\,[i(A_1 - A_2)\sin(\omega_1 t) + (A_1 + A_2)\cos(\omega_1 t)]$$

Substitute

$i(A_1 - A_2) = B$ and $A_1 + A_2 = C$

$$x(t) = \exp(-\gamma t)\,[B\sin(\omega_1 t) + C\cos(\omega_1 t)]$$

Introduce

$A = \sqrt{B^2 + C^2}$ and $\tan(\phi) = -\dfrac{C}{B}$

$$x(t) = A\exp(-\gamma t)\,\cos(\omega_1 t + \phi)$$

Damped oscillator moves at "frequency" smaller than undamped:

$$\omega_1 = \sqrt{\omega_0^{\,2} - \gamma^2} = \omega_0\sqrt{1 - \frac{\gamma^2}{\omega_0^{\,2}}}$$

For small damping γ expand in binomial series and retain only the first two terms

$$\omega_1 = \omega_0(1 - \frac{\gamma^2}{2\omega_0^{\,2}} + ...) \approx \omega_0(1 - \frac{\gamma^2}{2\omega_0^{\,2}})$$

and for small damping $\gamma \ll \omega_0$ and $\omega_1 \approx \omega_0$

Critically damped

$$\lambda_1 = \lambda_2 = -\gamma$$

Solution

$$x(t) = (A_1 + A_2)\exp(-\gamma t) = (B_1)\exp(-\gamma t)$$

This is not a general solution (it contains just one constant).
We can show that if
$x(t) = \exp(-\gamma t)$ than $x(t) = t\exp(-\gamma t)$ is also a solution.
Substitute

$\dot{x}(t) = \exp(-\gamma t) - \gamma t \exp(-\gamma t) = (1-\gamma t)\exp(-\gamma t)$

$\ddot{x}(t) = (1-\gamma t)(-\gamma)\exp(-\gamma t) - \gamma \exp(-\gamma t) = (\gamma^2 t - 2\gamma)\exp(-\gamma t)$

$[\gamma^2 t - 2\gamma + 2\gamma(1-\gamma t) + \omega_o{}^2 t]\exp(-\gamma t) = 0$

$[(\omega_o{}^2 - \gamma^2)t]\exp(-\gamma t) = 0$ Always satisfied.

Thus the general solution for a critically damped oscillator is:

$x(t) = (B_1 + B_2 t)\exp(-\gamma t)$

Overdamped

$\sqrt{\gamma^2 - \omega_0{}^2} < \omega_2$

$x(t) = \exp(-\gamma t)[A_1 \exp(\omega_2 t) + A_2 \exp(-\omega_2 t)]$

Energy considerations

Total energy:

$$E(t) = E(0) + \underbrace{W_f}_{\text{work performed by friction}}$$

Frictional force:

$f = -b\dot{x} = -bv$

$$W_f = \int_0^t f dx = \int_0^t f \frac{dx}{dt} dt = \int_0^t fv dt = \int_0^t -bv^2 dt$$

The rate of energy loss:

$$\frac{dE}{dt} = \frac{dW_f}{dt} = -bv^2$$

$$E(t) = K(t) + U(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

recall that

$x(t) = A\exp(-\gamma t)\cos(\omega_1 t + \phi)$

$\dot{x}(t) = -\omega_1 A \exp(-\gamma t)[\sin(\omega_1 t + \phi) + \frac{\gamma}{\omega_1}\cos(\omega_1 t + \phi)]$

Assume that the system is lightly damped ($\frac{\gamma}{\omega_1} \ll 1$), so we can neglect the second term.

Then:

$$E(t) = \frac{1}{2}A^2 \exp(-2\gamma t)[m\,\omega_1 \sin^2(\omega_1 t + \phi) + k\cos^2(\omega_1 t + \phi)]$$

Since light damping was assumed, $\omega_1^2 \approx \omega_0^2 = \frac{k}{m}$ and *E(t)* becomes

$$E(t) = \frac{1}{2}kA^2 \exp(-2\gamma t)$$

The initial energy

$$E_0 = \frac{1}{2}kA^2$$

and thus

$$E(t) = E_0 \exp(-2\gamma t)$$

(notice that the energy decays twice as fast as amplitude!)

The characteristic decay time (*E* decreases to *E/e*)

$$\frac{E_0}{e} = E_0 \exp(-2\gamma\tau)$$

$$2\gamma\tau = 1$$

$$\tau = \frac{1}{2\gamma} = \frac{2m}{2b} = \frac{m}{b}$$

Quality factor

$$Q = 2\pi \frac{\text{energy stored in the oscillator}}{\text{energy dissipated in one time period}}$$

Define P = power loss = rate of energy dissipation

since one time period $T_1 = \frac{2\pi}{\omega_1}$

the denominator can be written as

$$PT_1 = P\frac{2\pi}{\omega_1}$$

Thus

$$Q = \cancel{2\pi}\frac{E}{P\frac{\cancel{2\pi}}{\omega_1}}$$

Since $\frac{1}{\omega_1}$ - time necessary to complete 1 radian of oscillation, we can redefine Q as

$$Q = \frac{\text{energy stored in the oscillator}}{\text{average energy dissipated per radian}}$$

For a *lightly* damped oscillator Q can be calculated as follows:

$$E(t) = E_0 \exp(-2\gamma t)$$
$$\frac{dE}{dt} = -2\gamma E$$

Thus the energy dissipated in time Δt will be equal to

$$\Delta E = \left|\frac{dE}{dT}\right| \Delta t = 2\gamma E \Delta t$$

If we choose $\Delta t = \dfrac{1}{\omega_1}$ (time necessary to complete 1 radian)

$$Q = \frac{E}{\Delta E} = \frac{E}{2\gamma E / \omega_1} = \frac{\omega_1}{2\gamma}$$

For *light* damping $\omega_1 \approx \omega_0$ and thus

$$Q = \frac{\omega_0}{2\gamma}$$

Forced (driven) damped harmonic oscillator

Net force

$$F_{net} = \underbrace{F_s}_{\text{restoring elastic force}} + \underbrace{F_f}_{\text{dissipative friction force}} + \underbrace{F_d}_{\text{driving force}}$$

where: $F_s = -kx$; $F_f = -b\dot{x}$

$$F_{net} = m\ddot{x}$$
$$m\ddot{x} + b\dot{x} + kx = F_d$$

assume harmonic driving force

$$F_d = F_0 \cos(\omega t + \theta_0)$$

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t + \theta_0)$$

This is an inhomogeneous 2-nd order linear differential equation. Its solution is the sum of two parts, according to the following theorem:

> If X_i is a particular solution of an *inhomogeneous* differential equation, and X_n is a solution of a complementary *homogeneous* equation, then
>
> $X(t)=X_i(t)+X_h(t)$
>
> is a general solution.

From our previous considerations, the solution of the complementary *homogeneous* equation is given by one of the three equivalent forms:

$$x_h(t) = \exp(-\gamma t)\,[A_1 \exp(+i\omega_1 t) + A_2 \exp(-i\omega_1 t)]$$
$$x_h(t) = \exp(-\gamma t)\,[B \sin(\omega_1 t) + C \cos(\omega_1 t)]$$
$$x_h(t) = A_h \exp(-\gamma t) \cos(\omega_1 t + \phi_h)$$

For an *inhomogeneous* equation, let's postulate the following *particular* solution
$x_i(t) = A \cos(\omega t \pm \phi)$
and focus on the – sign solution.

$$x_i(t) = A \cos(\omega t - \phi)$$

$$\dot{x}_i(t) = -A\omega \sin(\omega t - \phi)$$
$$\ddot{x}_i(t) = -A\omega^2 \cos(\omega t - \phi)$$

Upon substitution
$$-mA\omega^2 \cos(\omega t - \phi) - bA\omega \sin(\omega t - \phi) + kA \cos(\omega t - \phi) = F_0 \cos(\omega t)$$
(for simplicity we assumed that $\theta_0 = 0$).

Recall that:

$$\boxed{\begin{aligned}\cos(\omega t - \phi) &= \cos\omega t \cos\phi + \sin\omega t \sin\phi \\ \sin(\omega t - \phi) &= \sin\omega t \cos\phi - \cos\omega t \sin\phi\end{aligned}}$$

thus
$$-mA\omega^2\,[\cos\omega t \cos\phi + \sin\omega t \sin\phi]$$
$$-bA\omega\,[\sin\omega t \cos\phi - \cos\omega t \sin\phi]$$
$$+kA\,[\cos\omega t \cos\phi + \sin\omega t \sin\phi] =$$
$$= F_0 \cos(\omega t)$$

This can be regrouped as

$$\cos\omega t[-mA\omega^2 \cos\phi + bA\omega \sin\phi + kA \cos\phi]$$
$$-\sin\omega t[-mA\omega^2 \sin\phi - bA\omega \cos\phi + kA \sin\phi]$$
$$= F_0 \cos(\omega t) + 0 \sin(\omega t)$$

Since the $\cos(\omega t)$ and $\sin(\omega t)$ coefficients on both sides of the equation have to be equal, we obtain the system of two equations:

$$(k - m\omega^2)\cos\phi + b\omega\sin\phi = \frac{F_0}{A}$$
$$(k - m\omega^2)\sin\phi - b\omega\cos\phi = 0$$

From the above it follows that:

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{b\omega}{k - m\omega^2} = \frac{b\frac{\omega}{m}}{\frac{k}{m} - \omega^2} = \frac{2\gamma\omega}{\omega_0^{\,2} - \omega^2}$$

thus

$$\sin\phi = \frac{2\gamma\omega}{\sqrt{(\omega_0^{\,2} - \omega^2)^2 + 4\gamma^2\omega^2}}$$
$$\cos\phi = \frac{\omega_0^{\,2} - \omega^2}{\sqrt{(\omega_0^{\,2} - \omega^2)^2 + 4\gamma^2\omega^2}}$$

Substistute these back to obtain

$$A = \frac{F_0 / m}{\sqrt{\left(\omega_0^{\,2} - \omega^2\right)^2 + 4\gamma^2\omega^2}}$$

Thus a particular solution of the *inhomogeneous* equation is:

$$x_i(t) = \frac{F_0 / m}{\sqrt{\left(\omega_0^{\,2} - \omega^2\right)^2 + 4\gamma^2\omega^2}}\cos(\omega t - \phi)$$

where

$$\phi = \tan^{-1}\frac{2\gamma\omega}{\omega_0^{\,2} - \omega^2}$$

The general solution is:

$$x(t) = x_h(t) + x_i(t) = \underbrace{A_h \exp(-\gamma t)\cos(\omega_1 t + \phi_h)}_{\text{transient term}} + \underbrace{\frac{F_0 / m}{\sqrt{\left(\omega_0^{\,2} - \omega^2\right)^2 + 4\gamma^2\omega^2}}\cos(\omega t - \phi)}_{\text{steady state term}}$$

Amplitude resonance

The amplitude of the particular solution reaches maximum when the driving force is equal to

$$\omega = \omega_r = \sqrt{\omega_0^2 - 2\gamma^2}$$

On resonance, the phase shift

$$\phi = \frac{\pi}{2}$$

Far below resonance

$\omega \ll \omega_r, \quad \phi \to 0$

Far above resonance

$\omega \gg \omega_r, \quad \phi \to \pi$

Energy resonance

$$x(t) = A\cos(\omega t - \phi)$$
$$v = \dot{x}(t) = -\omega A \sin(\omega t - \phi)$$
$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \phi)$$
$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t - \phi)$$

$$E(t) = K(t) + U(t) = \frac{1}{2}A^2[m\omega^2 \sin^2(\omega t - \phi) + k\cos^2(\omega t - \phi)]$$

Recall that per period:

$$\left\langle \cos^2(\omega t - \phi) \right\rangle = \left\langle \sin^2(\omega t - \phi) \right\rangle = \frac{1}{2}$$

Substitute $A(\omega)$ to *K(t)*

$$K(t) = \frac{1}{2}m\omega^2 \frac{F_0^2 / m^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2} \sin^2(\omega t - \phi)$$

$$\left\langle K(t) \right\rangle = \frac{1}{4}\frac{F_0^2}{m}\frac{\omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}$$

Kinetic energy resonance:

$$\frac{d\left\langle K(t)\right\rangle}{d\omega} = 0 \quad \text{for } \omega = \omega_0$$

Potential energy resonance

since $U(t) = \frac{1}{2}kx^2$, occurs at the same frequency as amplitude resonance $\omega = \sqrt{\omega_0^{\,2} - 2\gamma^2}$

Total energy resonance. Resonance peak width and Q

$$\left\langle E(t) \right\rangle = \frac{1}{2} A^2 m\omega^2 \left\langle \sin^2(\omega t - \phi) \right\rangle + \frac{1}{2} A^2 k \left\langle \cos^2(\omega t - \phi) \right\rangle$$

and after substituting A

$$\left\langle E \right\rangle = \frac{1}{4}\frac{F_0^{\,2}}{m}\frac{\omega^2 + \omega_0^{\,2}}{\left(\omega_0^{\,2} - \omega^2\right)^2 + 4\gamma^2\omega^2}$$

For very weak damping $\gamma \ll \omega_0$

$\omega^2 + \omega_0^{\,2} \approx 2\omega_0^{\,2}$

$\omega^2 - \omega_0^{\,2} = (\omega - \omega_0)(\omega + \omega_0) \approx 2\omega_0(\omega - \omega_0)$

$$\left\langle E(\omega) \right\rangle = \frac{1}{8}\frac{F_0^{\,2}}{m}\frac{1}{\left(\omega_0 - \omega\right)^2 + \gamma^2}$$

Lorentzian

Maximum at $\omega = \omega_0$

$$\left\langle E(\omega_0) \right\rangle = \frac{1}{8}\frac{F_0^{\,2}}{m}\frac{1}{\gamma^2}$$

The energy is equal ½ of its value at

$(\omega - \omega_0)^2 = \gamma^2$

or $\omega - \omega_0 = \pm\gamma$

Resonance peak width at half height

$\Delta\omega = 2\gamma$

Recall from previous considerations that

$Q = \frac{\omega_0}{2\gamma}$

This is the basis for determining Q from the energy resonance peak width:

$$Q = \frac{\omega_0}{\Delta\omega}.$$