06-713: Homework 6
Due October 31

1. Many experimental devices (for example, quartz crystal microbalances) involve components that are forced oscillators. If a forced oscillator has no damping, it is described by

$$m\frac{d^2x}{dt^2} + kx = F\cos(wt),$$

where m, k, F, and w are positive constants, x is the position of the oscillator. We will assume that $\sqrt{k/m} \neq w$.

(a) Show that a particular solution to the ODE above is $x_p(t) = \dfrac{F}{m(\frac{k}{m} - w^2)}\cos(wt)$.

(b) Find the general solution to the ODE above.

(c) If the ODE above is part of a boundary value problem with boundary conditions $x(0) = 0$ and $x(T) = 1$, does this boundary value problem have a unique solution?

2. Consider a rod of uniform cross-section made from a material with thermal conductivity k extending from $x=0$ to $x=L$. A series of heaters have been imbedded in the rod so that heat is generated inside the rod at rate $Q(x)=Q_0(x)$. Assume that there are no cross-sectional variations in the rod's temperature, that the rod is well insulated at $x=L$, and that the rod is held at constant temperature T_0 at $x=0$ and also along its length. In this case, the steady-state temperature profile of the rod is $T(x) = T_0 + y(x)$, where $y(x)$ satisfies $y'' - \mu^2 y = -Q_0(x)/k$ with $y(0) = 0$ and $y'(L) = 0$.

(a) Does the equation describing the steady-state temperature profile have a unique solution?

(b) Write down the Green's function for the ODE above use it to find an explicit expression for the temperature profile. You may write your solution in a form involving an integral. Describe why the solution you find is physically consistent.

(c) Would the Green's function you used in part (b) change if the internal heating was changed to $Q(x)=Q_0 \sin(x)$? What if the rod was insulated at both ends?

3. Greenberg Problem 6.4.8, parts (b) and (e) only.

4. [*Take home exam problem in 2006*] This problem deals with a more complicated population dynamics model than the simple models we considered in class. This model is defined by

$$\frac{ds}{dt} = D(s_0 - s) - \frac{1}{Y_s}\frac{\mu_{s,\max} s n_1}{K_s + s},$$

$$\frac{dn_1}{dt} = -Dn_1 + \frac{\mu_{s,\max} s n_1}{K_s + s} - \frac{1}{Y_p}\frac{\mu_{p,\max} n_1 n_2}{K_p + n_1},$$

$$\frac{dn_2}{dt} = -Dn_2 + \frac{\mu_{p,\max} n_1 n_2}{K_p + n_1}.$$

Only solutions with $s \ge 0$, $n_1 \ge 0$, and $n_2 \ge 0$ are physically interesting.

(a) In this part of the problem, fix (in consistent units) $D = 0.0625$, $s_0 = 0.5$, $\mu_{s,\max} = 0.25$, $K_s = 5\times10^{-4}$, $Y_s = 3.3\times10^{-10}$, $\mu_{p,\max} = 0.24$, $K_p = 4\times10^{-8}$, $Y_p = 1.4\times10^3$. Find all physically interesting fixed points of the population model and classify their stability. Combine this information with numerically calculated trajectories to describe what will happen for all possible physically interesting initial conditions.

(b) In the model above, *s* represents a food source for species 1. Describe the possible behaviors for the same population if the concentration of the food source is artificially held at a constant level, that is, the first equation above is replaced by $\frac{ds}{dt} = 0, s(t) = 0.5$.

Optional Problems

1. [*Take home exam problem in 2006*] Let *u* represent the electrostatic potential between two concentric metal spheres at radii R_1 and R_2, with $0 < R_1 < R_2$. If the potential of the inner sphere is held constant at V_1 volts and the potential of the outer sphere is held constant at $V_2 = 0$ volts, then the potential between the spheres satisfies

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} = 0 \text{ with } u(R_1) = V_1 \text{ and } u(R_2) = 0.$$

Suppose the $R_1 = 2$ cm, $R_2 = 10$ cm, and $V_1 = 110$ volts**.**

(a) Prove that this problem has a unique solution.
(b) Find the solution of this problem analytically.
(c) Solve the problem numerically using a shooting method. You must clearly define what numerical method you have used, define any choices that had to be made to apply this method (step sizes etc.), and show that your numerical solution converges to the analytic solution

2. For more practice with two dimensional nonlinear ODEs, choose examples from Greenberg Problem 7.4.2. In addition to finding and classifying the steady states, attempt to sketch the global phase portrait and then use numerical solutions of the ODEs to examine whether your sketch is correct.