<u>06-713: Homework 3</u>
Due Sept. 26

1. The matrix $C = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is symmetric. Show by direct calculation that the eigenvalues of *C* are real and that the eigenvectors form an orthogonal set. Also show how the vectors $(1,0,0)^T$, $(0,1,0)^T$, and $(0,0,1)^T$ can be expanded in terms of the eigenvectors of *C*.

2. Greenberg 11.3.8

3. Greenberg 11.3.11 (Rayleigh's quotient)

4. Greenberg 11.4.7 and 11.4.8

5. Assume that multinational companies based in the US, Japan, and Europe have total assets of \$5 trillion, with \$2 trillion in the US and \$3 trillion in Europe in 2007. Also assume that the distribution of assets in one year (in 2007 dollars) is related to the previous year's distribution by

$$US_{k+1} = \tfrac{1}{2}\, US_k + \tfrac{1}{2}\, J_k + \tfrac{1}{2}\, E_k$$
$$J_{k+1} = \tfrac{1}{4}\, US_k + \tfrac{1}{2}\, J_k$$
$$E_{k+1} = \tfrac{1}{4}\, US_k + \tfrac{1}{2}\, E_k$$

   Find the asset distribution in 2008 and 2020. What is the limiting distribution of assets in the distant future? Any model like this one is subject to uncertainties in its coefficients. How sensitive is this model to changes in its coefficients? To partially answer this question, make a plot of the long time limit of \$US as the cash flow from the US to Europe is increased from 1/8 to 1/2.

6. (*In class exam problem – 2006*) The matrix $\mathbf{A} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$ rotates a three dimensional vector by θ radians in the x-z plane without changing the vector's length.
   (a) State whether $\mathbf{A}\vec{x} = \vec{b}$ always has a unique solution and state why or why not.
   (b) Calculate the inverse of **A** (Hint: use geometric reasoning)
   (c) Calculate the norm of **A**.
   (d) Calculate the condition number of **A**. What does this result mean about the numerical stability of rotating a vector many times through many different angles by applying **A** to the vector?

7. (*In class exam problem – 2006*) Consider the function $f(x,y,z)=(x+3)^2+12(y-2)^2e^{(x+3)}+17z^2(y-2)$.

   (a) Show that this function has exactly one critical point.

   (b) Calculate the Hessian of $f(x,y,z)$ at the critical point. Is this matrix positive definite?

   (c) Calculate the Hessian of $-f(x,y,z)$ at the critical point of $f(x,y,z)$. Is this matrix positive definite?

   (d) What can you conclude from parts (b) and (c) about the critical point of $f(x,y,z)$?

Optional Problems

1. Greenberg 11.4.1(a)-(d)

2. (*In class exam problem – 2006*) Consider the system of linear difference equations $\vec{x}_{k+1}=\mathbf{A}\vec{x}_k$, where A is a diagonalizable square matrix.
   a. Describe the behavior of the solutions to these equations for large *k* if **A** is negative definite. State how (or if) this behavior depends on the initial iterate, $\vec{x}_0$.
   b. Describe the behavior of the solutions to these equations for large *k* is **A** is positive definite. State how (or if) this behavior depends on the initial iterate, $\vec{x}_0$.

3. *(In class exam problem – 2006)* **A** is a real 3x3 matrix with eigenvalues 2, 3, and 10 and corresponding eigenvectors $(1,1,1)^T$, $(1,1,0)^T$, and $(1,0,0)^T$.

   a. Find a matrix that has the properties listed above and state whether this is the only matrix that satisfies these properties.

   b. Is the matrix you defined in part (a) an orthogonal matrix? Is it a symmetric matrix? Is it skew symmetric?

   c. Are the eigenvectors of the matrix you defined in part(a) unique? If not, write down another set of eigenvectors for this matrix.

4. This problem deals with the vector space of polynomials of degree ≤ 2. An inner product for this vector space can be defined by $(f,g)=\int_0^\infty e^{-x}f(x)g(x)dx$. Construct a

complete orthogonal basis for this vector space and inner product. You may find it useful to know that $\int_0^\infty e^{-x} x^n \, dx = n!$ for integers $n > 0$. (This integral actually has a name - it is called the Gamma function, $\Gamma(n+1) = \int_0^\infty e^{-x} x^n \, dx$.)

5. The logistic map, $x_{n+1} = \lambda\, x_n\, (1-x_n)$ has stable periodic solutions for some values of λ. A periodic solution of period n is a set of iterates (x_i, x_{i+1}, … , x_{i+n}) that are repeated as the map is iterated. This map has a series of stable solutions of period 1, 2, 2^2, 2^3, 2^4, 2^5, … as l is increased from 3.4 to ~3.57. How many periodic solutions in this sequence can you find numerically?